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Vol. VI.

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Proceedings of the Indian Association for the Cultivation of Science.

CONDUCTED
BY
PROF. C. V. RAMAN, M.A., HON. D.Sc.

Vol. VII.
With Fourteen Plates.

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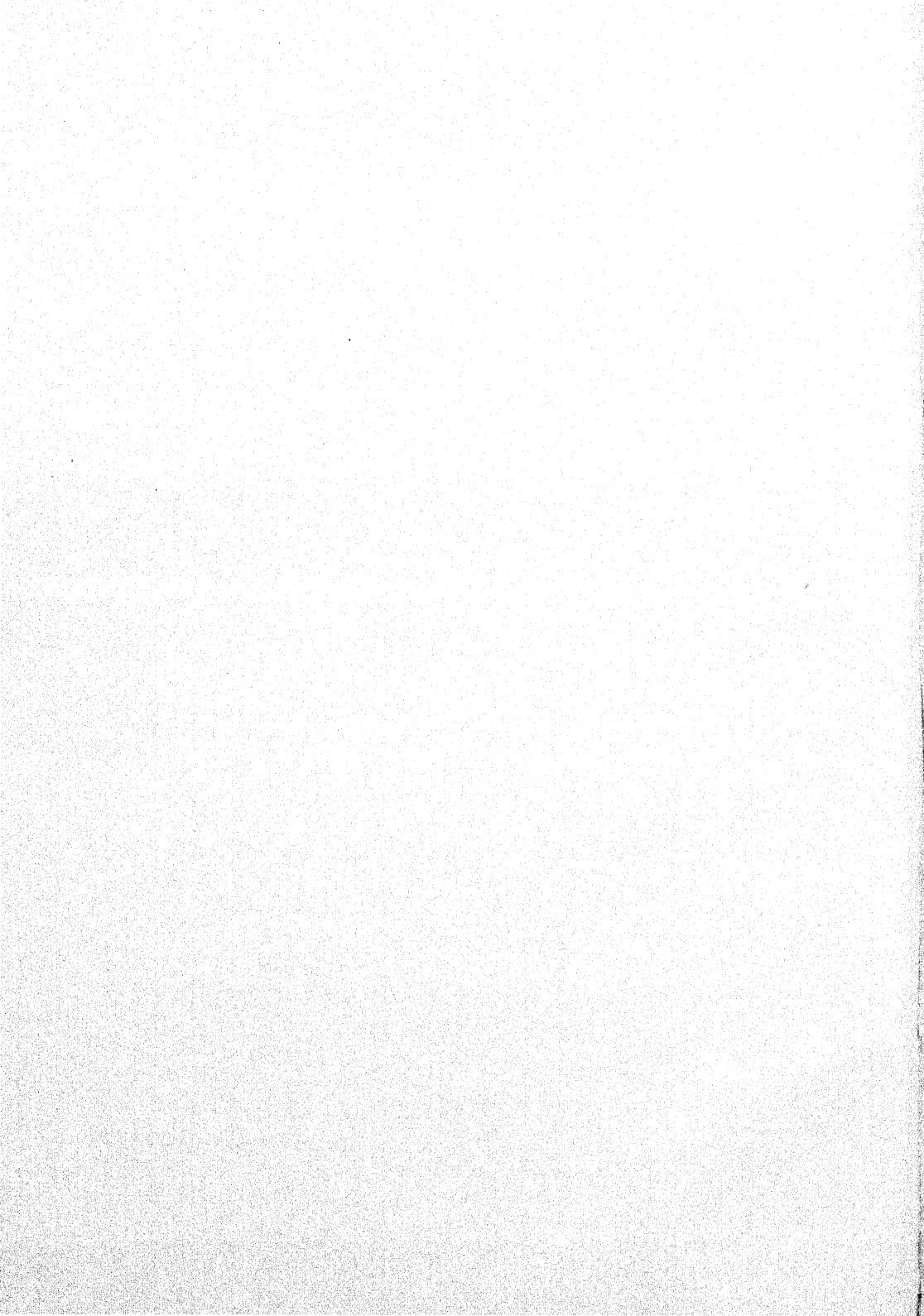
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I. On a New Geometrical Theory of the Diffraction-Figures observed in the Heliometer.

**By Sisir Kumar Mitra, M.Sc., Lecturer on Physical Optics in
the University of Calcutta.**

(Plate I).

CONTENTS.

- SECTION I.**—Introduction.
SECTION II.—Principle of the Geometrical Theory.
SECTION III.—Intensity in the Diffraction-Pattern in the two Principal
Directions.
SECTION IV.—General Configuration of the Pattern.
SECTION V.—The Determination of the Intensity of Illumination at any
Point in the Pattern.
SECTION VI.—Synopsis.

SECTION I.—INTRODUCTION.

A mathematical theory of the form of the diffraction figures of the Fraunhofer class due to a semi-circular aperture was first developed by Bruns* who gave the formulae for the intensity of the light at any point in the neighbourhood of the focus of a heliometer objective. These formulae, however, demanded a very large amount of arithmetical calculation before they could be of any practical use in the determination of the form of the diffraction pattern, and the further work involved in this appears to

* Bruns, "Über die Beugungsfiguren des Heliometer-Objectives," 'Astr. Nachr.', Bd. CIV, No. 2743.

have remained unattempted till 1909 when Everitt* succeeded in carrying out the numerical computations and finding the form of the contour-lines of equal illumination in the neighbourhood of the focus in the diffraction pattern. The process adopted by him was extremely elaborate, and involved the use of a Brunsviga calculator for part of the numerical work, of a Coradi co-ordinatograph, and of an integrator and planimeter for semi-graphical and mechanical integration. While Everitt's paper is of undoubted value in view of the painstaking thoroughness and accuracy of his work, the method adopted by him is unsatisfactory from a physical point of view, as it is essentially numerical in spirit, and leaves the reader without a clear conception of how precisely the peculiar configuration of the pattern arises from the semi-circular form of the aperture. Everitt's work was also of somewhat limited scope, as the calculation of intensities was confined to a relatively small region surrounding the position of the geometrical focus.

It will be shown in the present paper how the theoretical determination of the form of the diffraction figures in the heliometer can be carried out without appreciable loss of accuracy by a method which is geometrical in character and is far less laborious than that adopted by Everitt. Not merely is the method simpler, but it has also the advantage of enabling the form of the pattern to be readily determined for a region of any desired area surrounding the focus. The geometrical form of the pattern deduced by this method is in close agreement with experiment, and shows certain features at large angular deviations from the focus which were unsuspected by Everitt and which indeed make necessary a modification of the description of the diffraction-figures published by him.

To illustrate the novel features observed by the author, photographs of the diffraction pattern with relatively long exposures have been secured, one of which enlarged to a suitable size is reproduced in Fig. 1 (Plate). For purpose of comparison, a photograph taken with a comparatively small exposure is reproduced in

* Everitt, "Diffraction Figures due to the Heliometer," 'Proc. of the Royal Society of Lond.' Series A, Vol. 83; 1910.

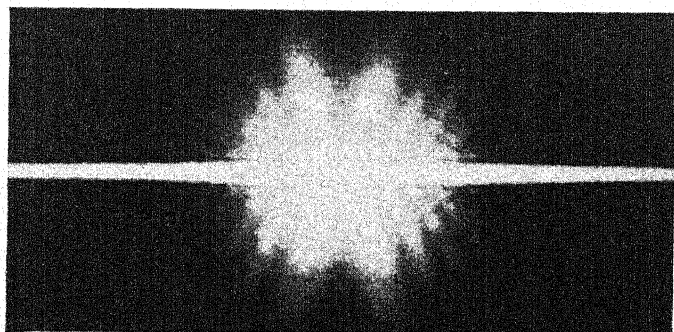


Fig. 1

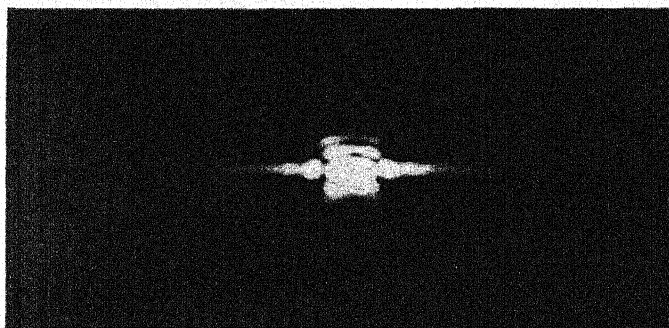


Fig. 2

Diffraction - Figures observed in the Heliometer.

Fig. 2, the picture in this case being very similar to those published in Everitt's paper.* It will be noticed that Fig. 1 (Plate) shows the prolongation of the transverse streamers as *concave* outwards, a feature which fails to appear in the relatively limited region covered by the picture in Fig. 2 (Plate). The work has also brought to light another important feature not mentioned by Everitt, that the fluctuations of intensity along the long horizontal ray that appears crossing the pattern decrease both relatively and absolutely at large angular deviations.

SECTION II.—PRINCIPLE OF THE GEOMETRICAL THEORY.

The ordinary treatment of diffraction phenomena, both of the Fresnel and of the Fraunhofer class, proceeds by expressing the effect at any point in the field in terms of a surface integral taken over the area of the aperture, in other words, as the resultant of a collection of secondary sources of light situated over the whole area of the boundary. A considerable simplification is, however, effected by transforming the surface integral into a line integral, in other words, by regarding the effect at any point in the field as due to a linear source of light situated along the boundary of the aperture, and the linear source in turn can be replaced by a finite number of point sources of light, generally two, sometimes three or more, having appropriate phases and situated at certain points on the boundary. The position and intensity of these point-sources on the boundary is generally not fixed, but varies with the direction of the diffracted light. In other words, corresponding to each point in the focal plane at which the diffraction pattern is formed, we have certain points on the boundary which principally contribute to the luminous effect at the point of observation, and the whole of the diffraction pattern may be simply regarded as an interference pattern due to a finite number of light-sources of *variable position* situated on the boundary of the aperture.

The foregoing method of regarding diffraction is of con-

* And also in a recent paper by Gordon, 'Proc. of the Phys. Soc. of Lond., October 1912.

siderable utility in forming a mental picture of the way in which the phenomena due to any aperture of arbitrary form arise, and explaining the geometrical relationship between the form of the aperture and the form of the diffraction pattern produced by it. This has been emphasised in a recent paper by me on large-angle diffraction by curvilinear boundaries.* So far from being merely a convenient mathematical fiction, the existence of sources of light situated at specific points on a curvilinear diffracting boundary may be directly verified by observation or photography, as has been shown in a recent paper.† For this purpose, the aperture may be viewed by the aid of the diffracted light only, admitted into an observing telescope through a small hole in a screen otherwise completely cutting off the light reaching the focal plane [as in the well-known method of the Foucault test]. When the luminous emission from the boundary of a semi-circular aperture is observed in the foregoing manner, the following phenomena are noticed : in general, *three* points on the boundary are seen to be luminous: two of equal but small intensity are situated on the two corners, and a third and more intense one is situated on the arc of the semi-circle at such point that the corresponding radius of the semi-circle is parallel to the line joining the centre of the focal plane with the orifice through which the diffracted light enters the observing telescope. There are, however, two exceptions to this general rule: when the orifice is situated at any point on the long horizontal ray in the diffraction-pattern running perpendicular to the diameter of the semi-circle, the whole of the diameter as well as the mid-point of the arc appear luminous. On the other hand, if the orifice in the focal plane is situated on a line drawn parallel to the diameter of the semi-circle, we have only two light sources visible which are situated respectively at the two extremities of the diameter and appear brilliantly luminous. The transition from the phenomena as observed in the two special portions of the orifice in the focal plane to that seen in the general case when it is placed in any arbitrary position takes place in a fairly sudden manner.

* S. K. Mitra, Phil. Mag., Sept. 1919.

† Dr. S. K. Banerjee, Phil. Mag., Jan. 1919.

SECTION III.—INTENSITY IN THE DIFFRACTION-PATTERN IN THE TWO PRINCIPAL DIRECTIONS.

The illumination at any point in the focal plane lying in directions parallel and perpendicular to the diameter of the semi-circular aperture may be very readily calculated as follows :

Case I : Parallel to the diameter.

In this case we may begin by dividing the semi-circle into narrow strips parallel to the diameter and adding up the effects of these strips.

To obtain the effect of one of these strips we observe that the expression for disturbance due to an elementary portion $dx dy$ of the strip (of width dy) may be written as

$$A' \sin \left(wt - \frac{2\pi}{\lambda} \frac{r}{f} x \right) dx dy,$$

(the X axis being taken parallel to the strips), where r is the distance of the point of observation from the centre of the pattern in the focal plane, f the focal length of the lens, and A' is a const. such that if I be the intensity at the centre of the pattern, $\sqrt{I} = A' \times$ area of the aperture. The above when integrated between the limits x_1 and x_2 , the co-ordinates of the two extremities of the strip gives us

$$\begin{aligned} & \frac{\lambda f A' dy}{2\pi r} \left[\cos \left(wt - \frac{2\pi}{\lambda} \frac{r}{f} x_2 \right) - \cos \left(wt - \frac{2\pi}{\lambda} \frac{r}{f} x_1 \right) \right] \\ &= \frac{2dy}{\pi R \delta} \left[\sin \left(wt - \frac{2\pi}{\lambda} \frac{r}{f} x_1 - \frac{\pi}{2} \right) + \sin \left(wt - \frac{2\pi}{\lambda} \frac{r}{f} x_2 + \frac{\pi}{2} \right) \right] \end{aligned}$$

where

$$\delta = \frac{2\pi}{\lambda} \frac{r}{f} R \text{ and } A' = \frac{2}{\pi R^2},$$

the intensity at the centre of the pattern being taken to be unity. R is the radius of the semicircular arc.

This shows that the effect of each strip may be replaced by that of two light-sources of equal strength and opposite phase situated at its extremities. (It will be noticed that the strength of the sources is independent of the length of the strip.) The effect of the whole aperture accordingly reduces to a linear distribution of light sources along the semicircular arc. If the aper-

ture had been a complete circle instead of being a semicircle, there would have been a precisely similar and symmetrical distribution along the other half of the circle. It follows from this that the resultant amplitude at any point due to the distribution of sources on the boundary of the semicircular aperture is half that for the case of the circular aperture, the intensity being one-fourth. *The positions of the maxima and minima of illumination along this direction are accordingly identical with those obtained in the well-known investigation for the case of a circular aperture.*

Case II : Perpendicular to the diameter.

In this case, the semicircular aperture may be imagined divided up into strips perpendicular to the diameter. Proceeding in the same manner as before, the effect of the whole aperture reduces to a linear distribution of sources along the straight and the curved parts of the boundary, which it is convenient to consider separately.

For the straight portion of the boundary we have a distribution of sources of amplitude

$$\frac{2dy}{\pi R\delta} \sin \left(wt - \frac{2\pi}{\lambda} \frac{r}{f} x_1 - \frac{\pi}{2} \right).$$

The X axis being taken parallel to the strips (i.e. perpendicular to the diameter), x_1 is constant, and the phases of the contributions of the different parts of the boundary are all the same. The effect due to the whole diameter is thus obviously

$$\frac{2 \times 2R}{\pi R\delta} \sin \left(wt - \frac{2\pi}{\lambda} \frac{r}{f} x_1 - \frac{\pi}{2} \right),$$

or taking the Y axis along the diameter (i.e. $x_1=0$), it is equal to

$$\frac{4}{\pi\delta} \sin \left(wt - \frac{\pi}{2} \right).$$

For the curved portion of the boundary we have a similar distribution of sources of amplitude

$$\frac{2dy}{\pi R\delta} \sin \left(wt - \frac{2\pi}{\lambda} \frac{r}{f} x_2 + \frac{\pi}{2} \right).$$

In this case, however, x_2 being different for different parts of the boundary, the phases of the contributions from different parts

would vary rapidly and would be stationary only at the mid-point of the arc. The resultant effect of the sources situated along the semicircular arc may be deduced at once from the consideration that if a precisely similar distribution of sources on a semicircular arc be placed symmetrically on the other side of the diameter, the joint effect of the two together should give us the well-known expression for the case of a complete circular aperture which is

$$4 \frac{J_1(\delta)}{\delta} \sin wt^*$$

Using semi-convergent expansions we have

$$\begin{aligned} J_1(\delta) &= \sqrt{\frac{2}{\pi\delta}} \left[\sin \left(\delta - \frac{\pi}{4} \right) + \frac{3}{8} \frac{\cos \left(\delta - \frac{\pi}{4} \right)}{\delta} \right] \\ &= \sqrt{\frac{2}{\pi\delta}} \sin \left(\delta - \frac{\pi}{4} + \frac{3}{8\delta} \right) \end{aligned}$$

approximately, and the expression for the disturbance due to the complete circular boundary is accordingly

$$\frac{4}{\delta} \sqrt{\frac{2}{\pi\delta}} \sin \left(\delta - \frac{\pi}{4} + \frac{3}{8\delta} \right) \sin wt$$

This disturbance may be resolved into two parts,

$$\begin{aligned} &\frac{2}{\delta} \sqrt{\frac{2}{\pi\delta}} \sin \left(wt - \delta + 3 \frac{\pi}{4} - \frac{3}{8\delta} \right) \\ \text{and} \quad &\frac{2}{\delta} \sqrt{\frac{2}{\pi\delta}} \sin \left(wt + \delta - 3 \frac{\pi}{4} + \frac{3}{8\delta} \right) \end{aligned}$$

each due to one-half of the circular boundary. The effect due to the distribution of sources situated along the semicircular arc in this case is thus equal to

$$\frac{2}{\delta} \sqrt{\frac{2}{\pi\delta}} \sin \left(wt - \delta + 3 \frac{\pi}{4} - \frac{3}{8\delta} \right).$$

* This is usually written $2 \frac{J_1(\delta)}{\delta} \sin wt$, taking the intensity at the centre of the pattern due to the *circular* aperture to be unity. Since we have taken the intensity at the centre of the pattern due to the *semicircular* aperture to be unity the expression is $4 \frac{J_1(\delta)}{\delta} \sin wt$.

The complete expression for the disturbance due to the semi-circular aperture is the sum of that due to the curved and straight parts of the boundary, that is equal to

$$\frac{2}{\delta} \sqrt{\frac{2}{\pi\delta}} \sin \left(wt + 3\frac{\pi}{4} - \delta - \frac{3}{8\delta} \right) + \frac{4}{\pi\delta} \sin \left(wt - \frac{\pi}{2} \right).$$

It is obvious that of these two terms, the latter varying as δ^{-1} ultimately predominates for increasing values of δ . The intensity in the diffraction pattern in the direction perpendicular to the diameter thus ultimately varies as δ^{-2} , while in a direction parallel to the diameter, it decreases much more rapidly in the ratio of δ^{-3} . Thus, we have in the diffraction pattern a long bright ray running perpendicular to the diameter of the semi-circle. Further, in the direction of the bright ray, not merely do the absolute intensities diminish, but the ratio of the intensities at maxima and minima also tends to approach unity, in other

			Author's Values.		Everitt's Values.	
			δ	H	δ	H
Centre	0.0	1.000	0.0	1.000
1st Min.	7.06	0.00897	7.1	0.009
1st Max.	9.65	0.0324	9.7	0.033
2nd Min.	13.40	0.00392	13.4	0.00397
2nd Max.	16.09	0.01050	16.0	0.01057
3rd Min.	19.69	0.00216	19.45*	0.00257*
3rd Max.	22.47	0.00506	22.45	0.00503
4th Min.	25.98	0.00137	25.95	0.00137
4th Max.	28.80	0.00294	28.85	0.00296
5th Min.	32.26	0.00094	32.25	0.00095
5th Max.	35.15	0.00191	35.15	0.00193
6th Min.	38.54	0.00069	38.50	0.00068
6th Max.	41.42	0.001342		
7th Min.	44.83	0.000535		
7th Max.	47.73	0.000980		
8th Min.	51.11	0.000422		
8th Max.	54.03	0.000763		
9th Min.	57.39	0.000340		
9th Max.	60.32	0.000599		
10th Min.	63.67	0.000284		
10th Max.	66.61	0.000469		
11th Min.	69.95	0.000240		
11th Max.	72.92	0.000402		

* Everitt's results seem to be in error.

words, the fluctuations of intensity along the bright ray tend to diminish both relatively and absolutely. The formula given above enables the intensity to be readily calculated for any value of δ with great ease. In the following table, several values and positions of the maxima and minima along the bright ray are shown, and alongside for comparison Everitt's values for the first few maxima and minima are also given. It will be seen that the same degree of accuracy as that obtained by Everitt is obtainable by this method with far less labour.

SECTION IV.—GENERAL CONFIGURATION OF THE PATTERN.

We now proceed to prove that (excluding the two principal directions) the whole effect of the aperture might be regarded as equivalent to three sources of appropriate phase and intensity two of which are situated at the two corners and the other on the

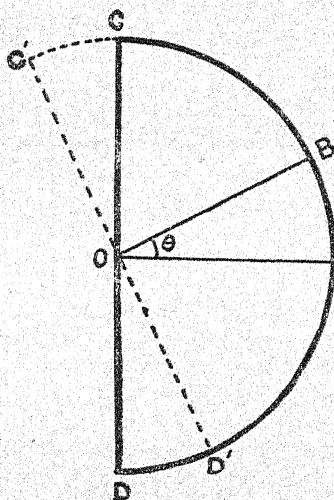


FIG. 3.

semicircular arc. The general configuration of the pattern will also be deduced on this basis.

Divide the semi-circular aperture into strips parallel to the line joining the point of observation to the centre of the pattern in the focal plane (as for instance parallel to OB in Fig. 3). As

before, the effect of all these strips can be reduced to a linear distribution of sources situated on the boundary of the aperture. The phases of the contributions from different parts of the straight portion of the boundary change regularly from C to D . We can accordingly replace the effect by a source situated at the centre O , or by two sources of opposite phase and equal amplitude situated at C and D . The phase of the resultant source at O being $-\pi/2$ (the elementary contributions being retarded by that amount on account of strip division), the phases of the component sources at C and D would be $-\delta \sin \theta$ and $\delta \sin \theta - \pi$ (the phases of the contributions from C and D are $-\delta \sin \theta$ and $\delta \sin \theta$ respectively, if the phase of contribution from O is taken to be zero, that from B being $-\delta$. $\theta = \pi/2 - \angle COB$). For the curved portion of the boundary, produce the semi-circle DBC to meet $C'D'$ drawn perpendicular to OB . We observe that the effect of the arc CBD is obviously equivalent to the effect of

(the semicircular arc $D'BC'$)—(the arc $C'C$) + (the arc DD').

The semicircular arc $C'B'D'$, we have already seen, can be replaced by a source of proper phase ($-\delta + \frac{3}{4}\pi$) and amplitude situated at B . The effect of the arcs CC' and DD' are replaceable by two sources at C and D respectively of equal amplitude (since the arcs are of equal length), whose phases are the same as the phases of the components at these points due to the straight portion of the boundary CD .* We thus have for the whole aper-

* This is evident from the following considerations: The amplitude of an elementary portion ds of the boundary being proportional to the width of the corresponding strip, is proportional to $ds \cos \psi$ (ψ = angle between the normal to ds and the length of the strip). The amplitude of an elementary contribution consequently changes from a maximum at B to zero value at C , while the phase advances from $-\delta$ to $-\delta \sin \theta$. If such a system of vibrations of continually decreasing amplitude and advancing phase is compounded, the phase of the resultant is approximately the same as that of the first vibration plus $\frac{\pi}{2}$. (This result is always true except when the phase of the first vibration passes through a maximum or minimum value. The above arguments are easily understood if the vibrations are compounded graphically. Cf. Preston, *Theory of Light* (third edition), page 250, fig. 120. The resultant effect due to the two arcs are therefore two sources at C and D of phases $-\delta \sin \theta + \pi/2$ and $\delta \sin \theta - \pi/2$ (unless C or D lie close to B , where the phase is a minimum). Remembering that the phases of the elementary portions of the arcs CC' and DD' are advanced and retarded respectively by $\pi/2$ (on

ture a source at B (due to the semicircular arc $D'BC'$), and a pair of sources at C and D (due both to the straight portion CD and the arcs CC' and DD'), their phases being

$$-\delta + \frac{3}{4}\pi \quad \text{at } B$$

$$-\delta \sin \theta \quad \text{at } C$$

$$\delta \sin \theta - \pi \quad \text{at } D.$$

(The source B is of course of variable position: its point of situation on DBC depending on the point of observation in the focal plane as explained before).

In order to find the nature of the diffraction pattern let us take any two of the sources and find out the positions of constant phase difference in the focal plane. Thus, for the sources C and B there would be maximum of illumination when

$$-\delta \sin \theta + \delta - \frac{3}{4}\pi = 2\pi, 4\pi, \text{ etc., i.e. } \delta - \delta \sin \theta = \text{const.}$$

This would give us in the focal plane a series of parabolas branching upwards, of common axis lying parallel to the diameter of the semicircle. Similarly the sources D and B would give us another set of parabolas, their axis the same as the former set, but the parabolas themselves branching downward.

$$\delta + \delta \sin \theta = \text{const.}$$

Also for the sources C and D , the condition of the maximum illumination is

$$\delta \sin \theta + \delta \sin \theta - \pi = 2\pi, 4\pi, \text{ etc., i.e. } \delta \sin \theta = \text{const.}$$

This gives us a number of straight lines in the focal plane running perpendicular to the diameter of the semi-circle, i.e. parallel to the horizontal bright ray. These features, viz. two sets of parabolas branching upwards and downwards, and a set of straight lines (a few of them) running perpendicular to the axis of the parabolas are shown in the drawings, Figs. 4 and 5. It will be seen that Fig. 4 closely reproduces the features appearing in the long-exposure photograph of the author in the Plate. (Fig. 1).

There is an alternative way of regarding the "interference"

account of strip division), and that the effect of CC' is to be subtracted, the phases of the resultants at C and D finally become $-\delta \sin \theta$ and $\delta \sin \theta - \pi$.

pattern in the focal plane which is both useful and instructive. The phase of the two sources at C and D being

$$-\delta \sin \theta \text{ and } \delta \sin \theta - \pi,$$

these when combined together give a source of fluctuating amplitude * at O proportional to $\sin (\delta \sin \theta)$, and phase $-\frac{\pi}{2}$.

The "interference" pattern in the focal plane can be regarded as due to the superposition of a source of fluctuating amplitude at O , and a source of variable position at B on the semicircular arc (the line OB being parallel to the line joining the point of observation in the focal plane to the centre of the pattern). Since the

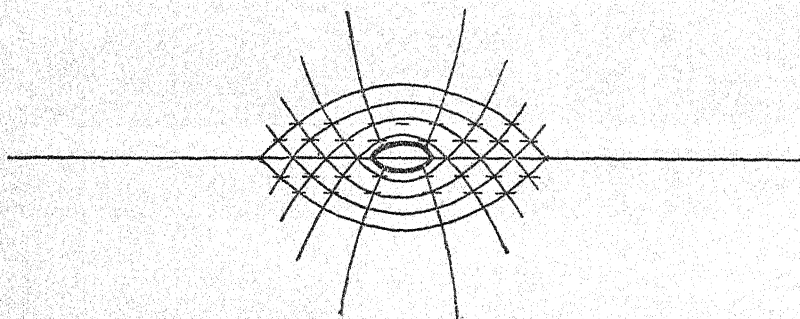


FIG. 4.

phases of these sources are independent of θ (i.e. direction of observation in the focal plane), the lines of equal phase will be circles round the centre of the pattern. The real maxima will however be at those positions in the focal plane where both O and B agree in phase, and at the same time the amplitude of the source O is a maximum (the amplitude of the other source being independent of θ).

* The part of the amplitude of the source at O due to the straight portion is as will be seen in the following section really proportional to $\frac{\sin (\delta \sin \theta)}{\delta \sin \theta}$, so that the positions of the maxima are slightly different from that given by $\delta \sin \theta = (2n+1)\frac{\pi}{2}$, when θ is small. This has been taken into account in drawing fig. 5. For small values of θ the other part of the amplitude (due to the arcs CC' and DD') is very small, so that the positions of maxima coincides with the maximum values of $\frac{\sin (\delta \sin \theta)}{\delta \sin \theta}$.

The first condition is given by

$$-\frac{\pi}{2} - \left(-\delta - \frac{3}{8\delta} + 3\frac{\pi}{4} \right) = 2\pi, 4\pi, \text{ etc.}$$

and the second by

$$\delta \sin \theta = \frac{\pi}{2}, 5\frac{\pi}{2}, 9\frac{\pi}{2}, \text{ etc.}$$

The first gives us a series of circles, and the second a series of parallel straight lines. The loci of maximum illumination lie on the intersections of these two sets of curves and are obviously parabolas. Fig. 5 shows the forms of the curves as obtained from the above considerations. The features appearing in the drawing

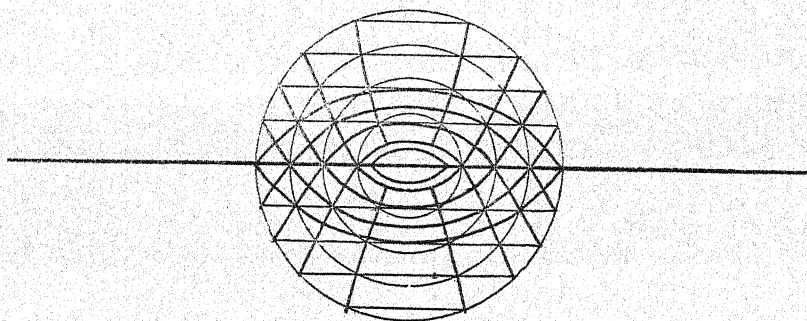


FIG. 5.

are also clearly noticeable in the long-exposure photograph in the Plate.

SECTION V.—THE DETERMINATION OF THE INTENSITY OF ILLUMINATION AT ANY POINT OF THE PATTERN.

The effect at any point in the focal plane being equal to the sum of two sources, one at O and another at a corresponding point B on the arc, we proceed to find the amplitude of these sources.

The amplitude of the source at B due to the semicircular arc $D'BC'$, we have already found to be

$$\frac{2}{\delta} \sqrt{\frac{2}{\pi\delta}},$$

that of the source at O being composed of two parts, one due to the straight portion CD and the other due to the arcs CC' and DD' , we can find their resultants separately.

The contribution from an elementary portion ds of CD being

$$\frac{2ds}{\pi R \delta} \cos \theta \sin \left(wt - \frac{2\pi s}{\lambda} \frac{\sin \theta}{f} - \frac{\pi}{2} \right)$$

its total effect is found by integrating the above expression between limits R and $-R$ (*vide* Fig. 3. $OC = OD = R$ and $s =$ distance of ds from O) to be

$$\frac{4}{\pi \delta} \cos \theta \frac{\sin (\delta \sin \theta)}{\delta \sin \theta} \sin \left(wt - \frac{\pi}{2} \right).$$

The resultant effect due to the arc CC' (or DD') can be found by dividing it into half-period "zones" and adding up the effects of all of these. This can be done in the following way:—

Beginning from B (fig. 3) divide the arc BC' into small parts such that the phases of the resultant contributions from successive parts (with reference to the point of observation in the focal plane) differ by π , that of the first one being the same as the resultant of the whole arc BC' . This is easily done by following a method exactly analogous to that employed by Schuster* in finding the resultant effect of a plane wave, and the total effect of the arc BC' , i.e. half the semicircular arc (except the first 'zone') can be expressed in the form

$$\frac{2}{\delta} \sqrt{\frac{2}{\pi \delta}} [(\sqrt{7} - \sqrt{3}) - (\sqrt{11} - \sqrt{7}) + (\sqrt{13} - \sqrt{11}) + \dots]$$

Knowing the effect due to half the semicircle to be

$$\frac{2}{\delta} \sqrt{\frac{2}{\pi \delta}} \times .5,$$

the effect of the first 'zone', since it is of opposite sign to that of the resultant of the rest of the arc, is easily seen to be

$$.6725 \times \frac{2}{\delta} \sqrt{\frac{2}{\pi \delta}}$$

* Schuster, Theory of Optics (2nd edition, 1909), page 95.

the sum of the above series within bracket being $\cdot 1725$. The effect of half the semicircular arc can thus be expressed in the form

$$\frac{2}{\delta} \sqrt{\frac{2}{\pi \delta}} (\cdot 6725 - \cdot 2908 + \cdot 2135 - \cdot 1771 \text{ etc.}).$$

By omitting the first term or the first two terms or the first three terms from the above series, the effect of corresponding portion of the arc as given in the following table can be found out:—

LENGTH OF THE ARC.	AMPLITUDE.
Half-semicircle $\delta - \delta \sin \theta = 0$	$\cdot 5 \times \frac{2}{\delta} \sqrt{\frac{2}{\pi \delta}}$
„ except 1st zone $\delta - \delta \sin \theta = \frac{3\pi}{4}$	$\cdot 1725 \times$ „
„ except first two zones $-\delta \sin \theta = \frac{7\pi}{4}$	$\cdot 1183 \times$ „
„ except first three „ $\delta - \delta \sin \theta = \frac{11\pi}{4}$	$\cdot 0925 \times$ „
etc.	etc.

If a curve be now plotted between different values of $\delta - \delta \sin \theta$ (corresponding to different lengths of the arc) and the corresponding effect in amplitude as found from the above table, we can obtain from that the effect in amplitude for *any* value of $\delta - \delta \sin \theta$, i.e. for *any* length of the arc CC' (or DD'). Designating the amplitude thus found by

$$\frac{2}{\delta} \sqrt{\frac{2}{\pi \delta}} \cdot f(\delta - \delta \sin \theta),$$

the expressions for the resultant sources at C or D due to the arcs CC' and DD' taking their proper phases into account as found in § 4 are at once obtained.

$$\frac{2}{\delta} \sqrt{\frac{2}{\pi \delta}} f(\delta - \delta \sin \theta) \sin (wt - \delta \sin \theta)$$

and

$$\frac{2}{\delta} \sqrt{\frac{2}{\pi\delta}} f(\delta - \delta \sin \theta) \sin (wt + \delta \sin \theta - \pi).$$

These when combined give us a resultant source

$$2 f(\delta - \delta \sin \theta) \frac{2}{\delta} \sqrt{\frac{2}{\pi\delta}} \sin \left(\delta \sin \theta \right) \sin \left(wt - \frac{\pi}{2} \right)$$

at O .

We thus have the complete expression for amplitude for any point in the focal plane:

$$\frac{2}{\delta} \sqrt{\frac{2}{\pi\delta}} \sin \left(wt + 3\frac{\pi}{4} - \delta - \frac{3}{8\delta} \right) + \left[\frac{4}{\pi\delta} \cos \theta \frac{\sin (\delta \sin \theta)}{\delta \sin \theta} + 2f(\delta - \delta \sin \theta) \frac{2}{\delta} \sqrt{\frac{2}{\pi\delta}} \sin (\delta \sin \theta) \right] \sin \left(wt - \frac{\pi}{2} \right).$$

In the following table are given the values of intensity H for various values of θ and δ , in the focal plane together with Everitt's values for comparison. The latter were obtained from his contour diagram of equal intensity.

The agreement can be seen to be satisfactory.

$$\delta = 7.01$$

	θ	0°	$9^\circ 42'$	33°	50°	63°	73°	90°
Author ..	H	.0092	.0050	.0163	.022	.007	.0033	.0000
Everitt ..	H	.0098	.004	.015	.015	.006	.002	.0000

$$\delta = 10.4$$

	θ	0°	$13^\circ 48'$	32°	$41^\circ 48'$	$55^\circ 12'$	70°	90°
Author ..	H	.0291	.0068	.0007	.0040	.0045	.0015	.00012
Everitt ..	H	.0291	.006	.001	.004	.004	.001	.00011

$$\delta = 13.32$$

	θ	0°	15°48'	28°6'	39°42'	50°48'	60°18'	69°30'	80°24'	90°
Author	H	.0039	.0021	.0010	.00038	.0020	.0022	.0012	.00036	.0000
Everitt	H	.004	.002	.001	.0005	.002	.002	.001	.0002	.0000

$$\delta = 16.47$$

	θ	0°	5°30'	21°48'	31°24'	48°48'	56°	63°42'	90°
Author	H	.0102	.0059	.00052	.0011	.00049	.0010	.0011	.0000
Everitt	H	.0103	.006	.0005	.001	.0005	.001	.001	.0000

SECTION VI.—SYNOPSIS.


1. Photographs of the diffraction pattern due to a heliometer objective with relatively long exposures have been secured which show certain features hitherto unsuspected by previous observers. The pattern roughly consists of two sets of parabolas (with a common axis, at right angles to the bright horizontal ray) branching in opposite directions. These features do not clearly appear in the relatively small region covered by the pictures of previous workers. Another important feature not mentioned by Everitt, who has made a detailed study of the diffraction figures, is that the fluctuations of intensity along the long bright ray that appears crossing the pattern decrease both relatively and absolutely at large angular deviations.

2. A new geometrical theory has been developed by means of which the light intensity at any point in the focal plane is considered as merely due to the interference of two or three light sources situated at definite points on the boundary. This is done by a simple transformation of the ordinary expression for the light vector taken as an integral over the surface of the aperture, into a line integral taken round the boundary, which latter is

finally reduced to two or three sources of proper phase and amplitude situated on the boundary of the aperture. This method greatly facilitates numerical computation of the light intensity for small as well as for very large angles of diffraction (the latter being almost impossible to obtain by the ordinary treatment), and gives results, which are for all practical purposes, as accurate as that obtained by elaborate mathematical formulae. It has also the additional advantage of enabling the form of the pattern to be readily determined for any desired area surrounding the focus, and the geometrical form thus deduced is in close agreement with that determined experimentally.

3. The writer has made preliminary observations on the diffraction figures due to apertures greater and less than a semicircle respectively, and has found that in these cases also the form of the pattern can be deduced from the above simple geometric considerations.

The investigation was carried out in the Palit Laboratory of Physics, and the writer wishes to thank Prof. C. V. Raman for his helpful interest during the progress of the work.



II.—Experiments with Mechanically-Played Violins.

By Prof. C. V. Raman.

(Plate II.)

CONTENTS.

SECTION	I.—Introduction.
SECTION	II.—Description of Mechanical Player.
SECTION	III.—Variation of Bowing Pressure with the Position of the Bowed Region.
SECTION	IV.—Relation between Bowing Speed and Bowing Pressure.
SECTION	V.—Variation of Bowing Pressure with Pitch.
SECTION	VI.—Effect of Muting on the Bowing Pressure.
SECTION	VII.—Other Applications of the Mechanical Player.
SECTION	VIII.—Synopsis.

SECTION I.—INTRODUCTION.

In the first volume (recently published *) of my monograph on the theory of the violin family of instruments, I have discussed on mechanical principles, the relation between the forces exerted by the bow and the steady vibration maintained by it, and the conditions under which the bow is capable of eliciting a sustained musical tone from the instrument. An experimental test of the results indicated by the theory on these points would obviously be of interest. Especially is this the case, as the analysis shows that the yielding of the bridge and the communication of energy from the strings through their supports into the instrument and thence into the air, play a very large part in determining the

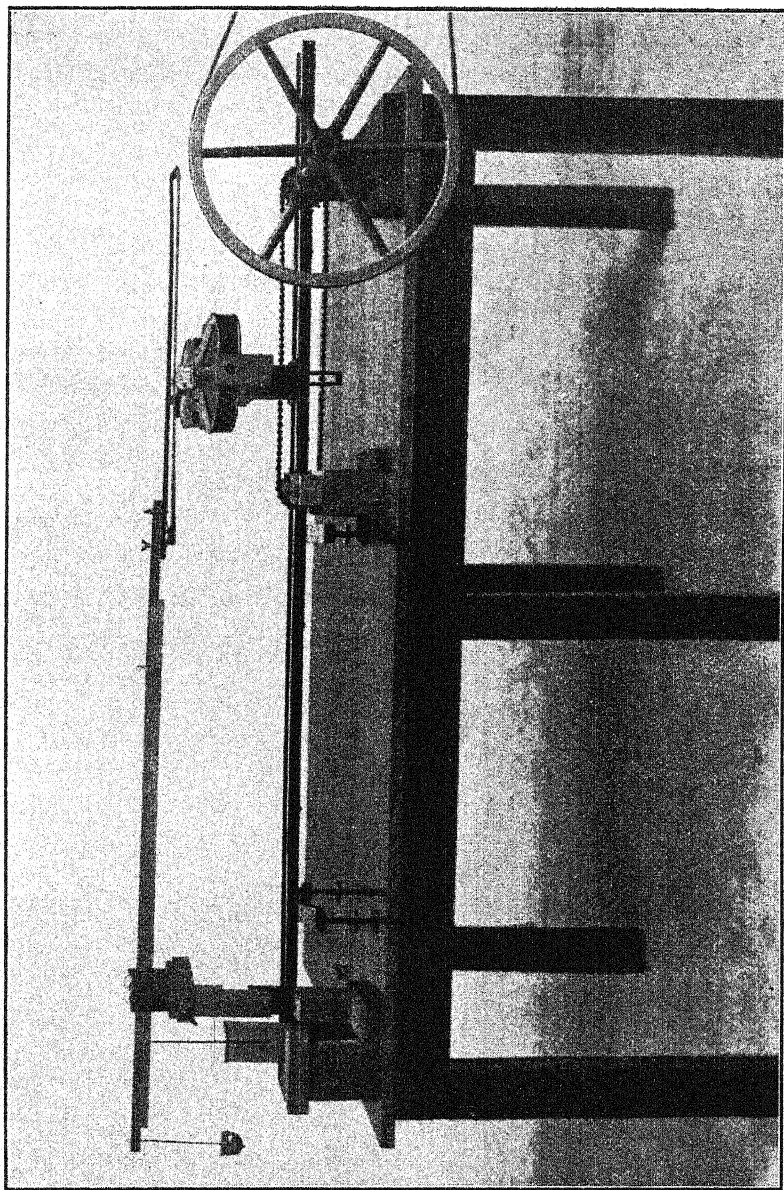
* Bulletin No. 15 of the Indian Association for the Cultivation of Science, 1918, pages 1-158.

magnitude of the forces required to be exerted by the bow. An experimental study of the mechanical conditions necessary for obtaining a steady musical tone could thus be expected not merely to throw light on the *modus operandi* of the bow but also to furnish valuable information regarding the instrument itself, its characteristics as a resonator and the emission of energy from it in various circumstances. Further, a study of the kind referred to could be expected also to furnish illustrations of the physical laws underlying the technique of the violinist and to put these laws on a precise quantitative basis. The experiments described in the present paper were undertaken with the objects referred to above, and the description of the results now given in these Proceedings is preliminary to a more exhaustive treatment of the subject which it is proposed to give in the second volume of my monograph under preparation for publication as a Bulletin of the Association.

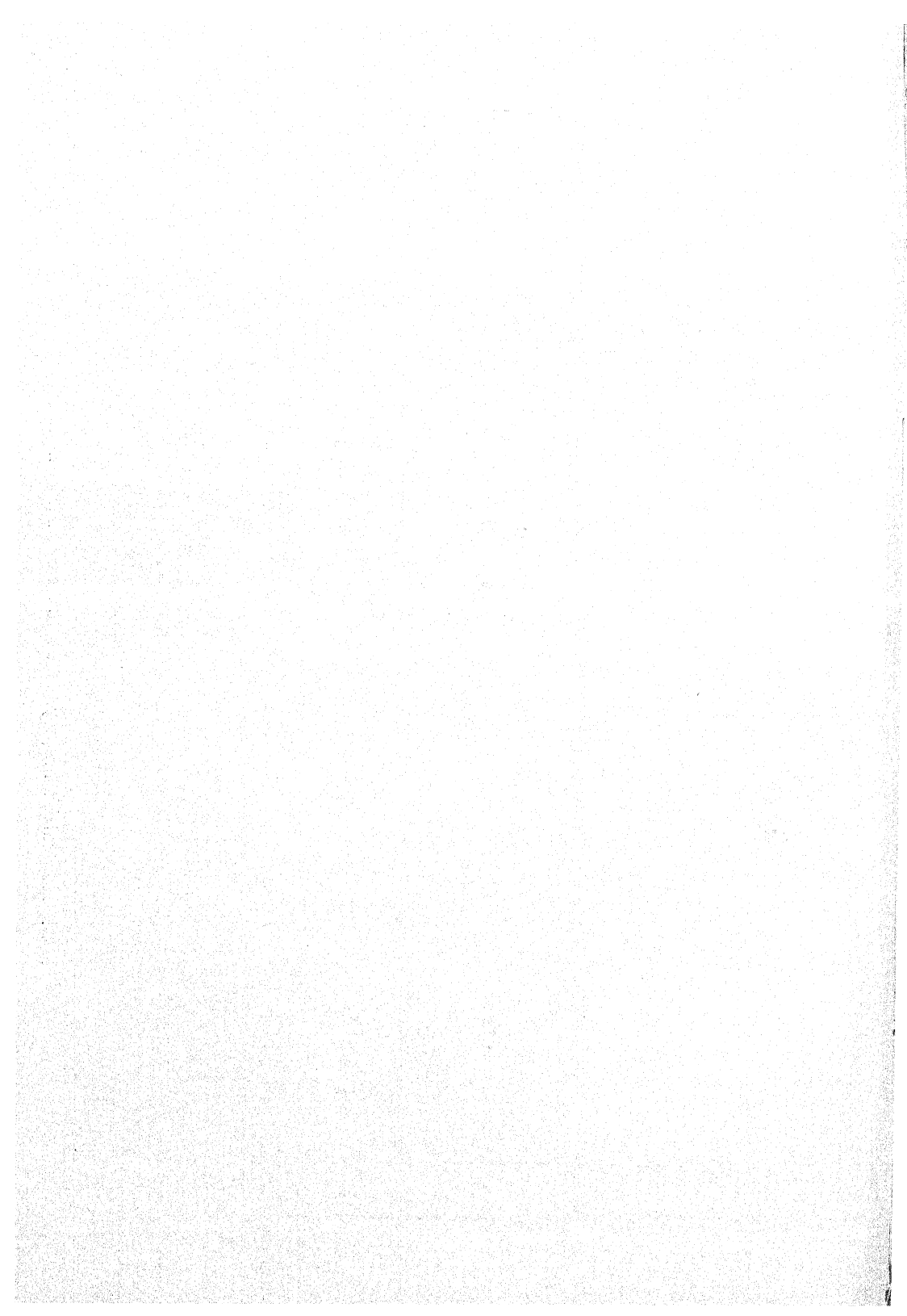
SECTION II.—DESCRIPTION OF MECHANICAL PLAYER.

As the object of the work was to elucidate the theory of production of musical tone from instruments of the violin family, it was decided that the experimental conditions should approximate as closely as practicable to those obtaining in ordinary musical practice. The general principle accordingly held in view in designing the mechanical player was to imitate the technique of the violinist as closely as possible. There was also another reason for adopting this course. It is well known that the bowing of a stringed instrument so as to elicit a good musical tone is an art requiring much practice for its perfect accomplishment. The performance of the same task by purely mechanical appliances under such conditions as would permit of accurate measurements of the pressure and speed of bowing and the discrimination by ear of the effect of varying these factors obviously involves difficulties which it was thought would be best surmounted by imitating the violinist's handling of the bow as closely as the mechanical conditions would permit. A mechanical player designed on this general idea which has fulfilled the requirements of the work is illustrated in Plate II.

As can be seen from the photograph, a violin and a horse-hair bow of the ordinary type were used in the mechanical player.



A Mechanical Violin-player for Acoustical Experiments.



Instead, however, of moving the bow to and fro, it was found a much simpler matter from the mechanical point of view to keep the bow fixed and to move the violin to and fro with uniform speed. This was arranged by holding the violin lightly fixed in a wooden cradle, the points of support being the neck and the tail piece of the violin as in the ordinary playing of the instrument. The cradle was mounted on a brass slide which moved to and fro noiselessly on a well-oiled cast-iron track. The slide received the necessary movement forward and backward from a pin carried by a moving endless chain and working in a vertical slot carried by the slide. The chain was kept in motion by the rotation of one of the two hubs between which it was stretched, this hub being fixed on the same axis as the driving wheel seen in the Plate.*

The apparatus was driven by a belt running over a conical pulley which in its turn was driven by a belt passing over the pulley of a shunt-wound electric motor controlled by a rheostat which was allowed to run without any load except the apparatus. Using the rheostat and a Weston Electrical Tachometer, a very constant speed could be maintained during the experiments. Different speeds of motion of the slide carrying the violin were obtained by putting the driving belt of the apparatus on to different parts of the conical pulley, or by adjusting the rheostat.

The mounting of the bow required special attention in order to ensure satisfactory results. As is well known, the violinist in playing his instrument handles the bow in such manner that when it is applied with a light pressure, only a few hairs at the edge touch the string. The bow is held carefully balanced in the fingers of the right hand, the necessary increases or decreases in the pressure of bowing being brought about by increase or decrease of the leverage of the fingers. The suppleness of the wrist of the player and the flaccidity of the muscles of the fore-arm secures the necessary smoothness of touch. These features are carefully imitated in the mechanical player. The violin-bow is held fixed at the end of a wooden lath, an adjustment being provided so that fewer or more hairs of the bow may be made to touch the string

* The whole of the apparatus was improvised in the laboratory from such materials as were to hand. The slide and cast-iron track were parts of a disused optical bench. The chain and hubs were spare parts purchased from a cycle-dealer. The ball-bearing of the axle of the lever (referred to below) was also part of a cycle. The other fittings were made up in the workshop.

of the violin. The lath itself is balanced after the manner of a steelyard, the axis of the lever being mounted on ball-bearings so as to secure the necessary solidity combined with freedom of movement. The weight of the bow is balanced by a load hung freely near the end of the shorter arm of the lever. The axis of the lever can be raised or lowered to the proper height above the violin such that when the hairs of the bow touch the string, they are perfectly parallel to the cast-iron track along which the violin slides. This is of great importance in order to obtain steady bowing, as otherwise the bow would swing up and down with the movement of the violin along the track, and its inertia would result in a variation of the pressure exerted by it. Any residual oscillations of the bow due to the elasticity of the lever or imperfection in the adjustment referred to above are checked by the damping arrangement shown in the Plate. A wire with a number of horizontal disks attached to it at intervals is hung freely from the shorter arm of the lever and dips inside a beaker of water or light oil. This effectually prevents any rapid fluctuations in the pressure of the bow and ensures a smooth movement. The pressure exerted by the bow on the string can be varied by moving a rider along the longer arm of the lever which is graduated. An adjustment is provided by which the block carrying the axis of the lever can be moved by a screw perpendicular to the track, and the distance from the violin-bridge of the point at which the bow touches the string may thus be expeditiously altered.

It will be noticed that with the arrangements described above, the pressure exerted by the bow on the string of the violin would not be absolutely constant throughout, but would vary somewhat as the violin moves along its track from the point nearest to the point furthest from the axis of the lever. This is not however a serious difficulty as the lever is fairly long and the variation of pressure is thus not excessive. Further, the observations of the character of the tone are always made for a particular position and direction of movement of the violin and no ambiguity or error due to the cause referred to above arises.

The speed of bowing may be readily determined from the readings of the Electrical Tachometer or directly by noting on a stop-watch the time taken for a number of strokes to and fro of the violin on its track.

SECTION III.—VARIATION OF BOWING PRESSURE WITH THE POSITION OF THE BOWED REGION.

One of the well-known resources of the violinist is to bring the bow nearer to or to remove it further away from the bridge of the violin, the extreme variation in the position of the bow being from about $\frac{1}{8}$ th to about $\frac{1}{2}$ th of the vibrating length of the string from the bridge. In a recent paper on "The Partial Tones of Bowed Stringed Instruments" published in the "Philosophical Magazine" (November 1919), I have discussed in some detail the changes in the amplitudes and phases of the various partials brought about by these changes in the position of the bowed region. In all the cases of musical interest within these limits, the mode of vibration of the string is practically the same as in the principal Helmholtzian type* in regard to the first three partial components, but differs from it in respect of the higher components to an extent depending on the removal of the bow from the bridge. The ratios of the amplitudes and the relative phases of the first three partials remain practically the same throughout the range, the actual amplitudes for a given speed of the bow varying inversely as the distance of the bowed point from the bridge. The amplitudes of the fourth, fifth and higher partials vary in a similar way with the position of the bowed point provided this is not too far from the bridge, but deviate from this law more and more as the bow is removed further and further from the bridge. The net effect of bringing the bow nearer the bridge (its speed remaining constant) is greatly to increase the intensity of the tone of the instrument, and to make it somewhat more brilliant in character, as is of course well known. Simultaneously with these changes, the pressure with which the bow is applied has to be increased. The mechanical player described above may be used to find experimentally the relation between the bowing pressure and the position of the bow under these conditions.

The graphs in Fig. 1 (thin lines) represent the results obtained with the player on the D-string of the violin. A few words of explanation are here necessary. As a finite region of the bow is in contact with the string, it is not possible to specify the position

* The principal Helmholtzian type is the mode of vibration in which the time-displacement graph of every point on the string is a simple two-step zig-zag.

of the bow by a single constant. Accordingly, the positions of the inner and outer edges of the region of contact were noted in the observations. The graph therefore shows two curves connecting the positions of the two edges of the bowed region with the

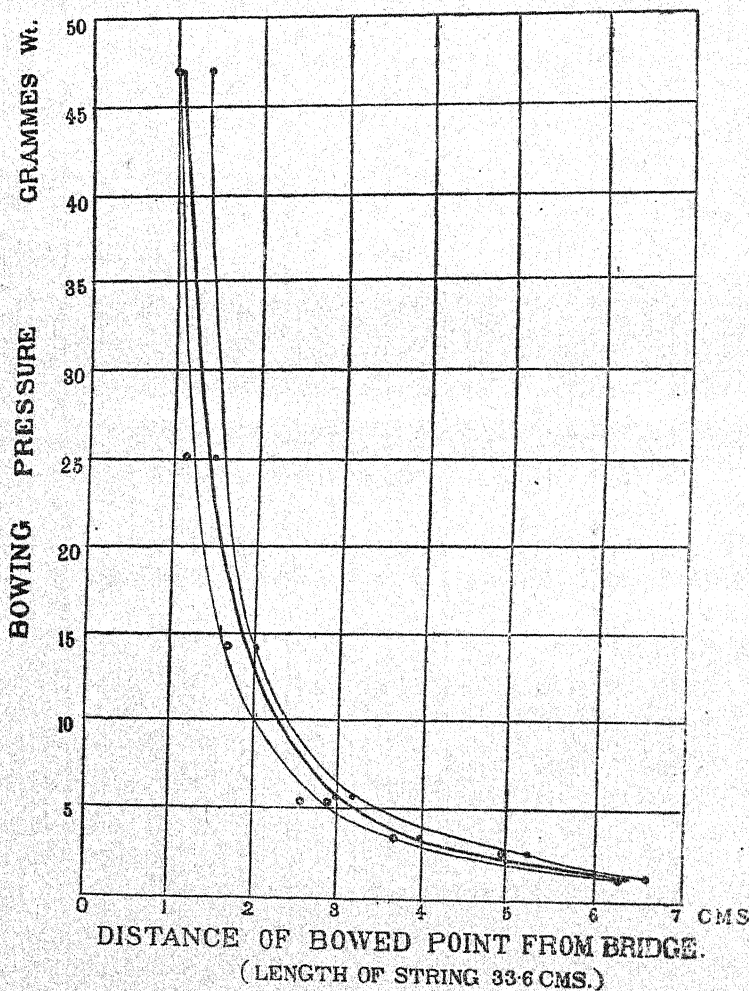


FIG. 1.

magnitude of the bowing pressure. The ordinates of the graphs represent the values of the *minimum* bowing pressure found necessary to elicit a full steady tone with pronounced fundamental. [For bowing pressures smaller than this minimum, the fundamental

falls off in intensity, and the prominent partial becomes its octave or twelfth. In certain cases, as for example near the wolf-note pitch, we get 'cyclical' or 'beating' tones.] It will be noticed from the graphs that the bowing pressure necessary increases with great rapidity when the bow is brought near the bridge.

The curve in Fig. 1 (heavily drawn) lying between the experimental graphs is a representation of the algebraic curve $x^2y = \text{constant}$. (The constant was, of course, suitably chosen.) It will be noticed that the graph follows the trend of the experimental values quite closely. In other words, we may say that in the cases studied, the bowing pressure necessary varies practically in inverse proportion to the *square* of the distance of the bow from the bridge. It may be readily shown that this is the result to be expected from theory. In my monograph,* I have shown that the minimum bowing pressure P is given by the formula

$$P = \frac{P_A' - P_A}{\mu - \mu_A}$$

where P_A' is the maximum value at *any* epoch of the series

$$\sum_{n=1}^{n=\infty} k_n B_n \frac{\sin \left(\frac{2n\pi t}{T} + e_n + e_n' \right)}{\sin \frac{n\pi x_0}{L}},$$

and P_A is the value of the series at the epoch at which the bowed region of the string slips past the bow. μ is the statical coefficient of friction, and μ_A is the dynamical coefficient of friction during the epoch of slipping. B_1, B_2 , etc. are the amplitudes of the partial vibrations of the string, k_1, k_2 , etc. are numerical constants for the respective partials depending on the instrument, the mass, length and tension of the string, and x_0 is the distance of the bowed point from the end of the string. We have already seen that amplitudes B_n of the first few partials for a given speed of bowing vary in inverse proportion to the distance of the bow from the bridge, and their relative phases remain unaltered. In respect of these partials, the factor

$$1/\sin \frac{n\pi x_0}{L}$$

* Bulletin No. 15, pages 73 to 75.

also varies practically in inverse proportion to x_0 , so long it is a small fraction of l .

To effect a simplification, we may proceed by ignoring the influence of all the partial vibrations except the first few, an assumption which is justifiable in the case under consideration, as by far the greater proportion of the energy of violin-tone is confined to the first few partials. Further, we may for simplicity, treat the difference $(\mu - \mu_A)$ between the statical and dynamical coefficients of friction as practically a constant quantity. This will not introduce serious error, provided the speed of the bow is not very small. For, if the slipping speed be fairly large, any changes in it due to change of the position of the bowed point would not seriously alter the dynamical coefficient of friction. On these simplifying assumptions, it will be seen from the formulæ given above that, within the limits considered, the minimum bowing pressure should vary in inverse proportion to the *square* of the distance of the bow from the bridge, exactly as found in experiment. This relation would, of course, cease to be valid when the bowed point is removed too far from the bridge or when the speed of the bow is very small.

SECTION IV.—RELATION BETWEEN BOWING SPEED AND BOWING PRESSURE.

The changing of the speed of the bow is another of the well-known resources of the violinist. The principal effect of this is to alter the intensity of tone. *Pari passu* with the change of speed of the bow, other things remaining the same, the violinist has to alter the pressure of the bow. The relation between these may be readily investigated with the mechanical player. The experimental results for the D-string and for a particular position of the bowed point are shown in Fig. 2.

The graph shows the following features: (1) for very small bowing speeds, the bowing pressure tends to a finite minimum value; (2) the increase of bowing pressure with speed is at first rather slow; (3) later, it is more rapid, the pressure necessary increasing roughly in proportion to speed, and for large amplitudes of vibration possibly even more than in proportion to the speed of the bow.

The foregoing results are, broadly speaking, in agreement with what might be expected on theoretical grounds.* This can be seen from the formula for the bowing pressure referred to in Section III. With increasing speed of the bow, the amplitudes B_n of the partial vibrations increase in proportion, so that if the difference $\mu - \mu_A$ between the statical and dynamical coefficients of friction be

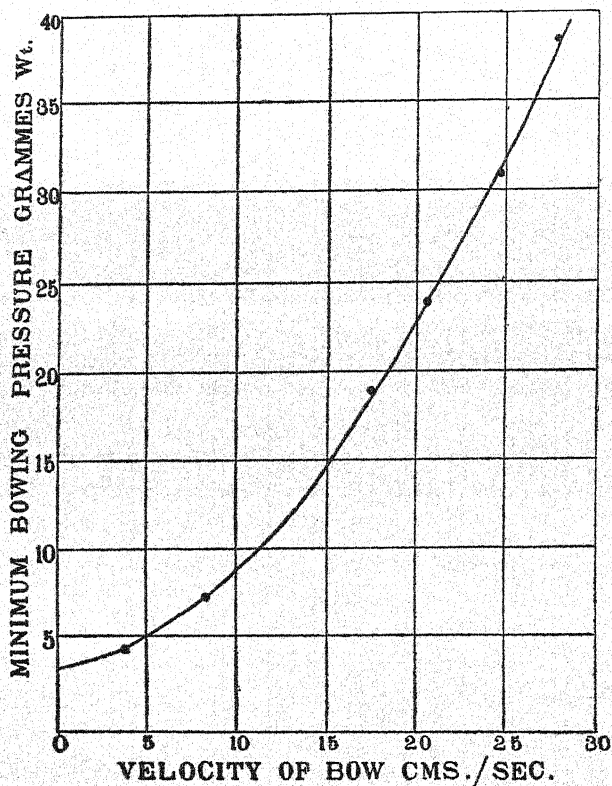


FIG. 2.

regarded as a constant, the bowing pressure necessary should vary directly as the speed of the bow. For very small speeds of the bow, however, it is not correct to take $\mu - \mu_A$ as constant, and it would be nearer the mark for such speeds to take $\mu - \mu_A$ as proportional to the velocity of slip, that is, as proportional to the speed of the bow. Thus, for very small speeds of the bow, the pressure neces-

* Bulletin No. 15, pages 151-153.

sary should be nearly independent of the speed, that is, should converge to a finite minimum speed. For larger speeds of the bow, it would be correct to take $\mu - \mu_A$ as constant and the bowing pressure should then vary proportionately with the speed. For very large speeds, the theory of small oscillations would no longer be applicable, and the quantities k_1, k_2, k_3 , etc. might increase with the speed of the bow. For such large speeds, the bowing pressure necessary might increase more than in proportion to the speed of the bow.

A more precise discussion of the experimental results would be possible on the basis of quantitative data as to the manner in which the coefficient of friction between rosined horse-hair and catgut varies with the velocity of slip at different pressures.

SECTION V.—VARIATION OF BOWING PRESSURE WITH PITCH.

The pitch of violin tone depends on (1) the linear density of the bowed string, (2) its length, and (3) its tension, and may be varied by varying any one or other of these factors. In practice, the violinist varies the pitch by (1) altering the vibrating length by "stopping" down the string on the fingerboard, or (2) by passing from one string to another. The mechanical player may be used to investigate the dependence of bowing pressure upon pitch when the latter is varied in any of the ways that may be suggested. Obviously, the sequence of phenomena observed would not be exactly the same for the four strings of the violin as these are of different densities and tension, communicate their vibrations to the body of the instrument at different points of the bridge and also vibrate in considerably different planes relatively to the bridge and belly when excited by the bow in the usual way. In the experimental work now to be described, a particular string of the violin, e.g. the 4th or G-string, was used, and the pitch was varied as in the ordinary playing of the instrument by 'stopping' the string at different points. This was arranged by clamping the string down to the fingerboard, with a light but strong brass clamp shaped like an arch which could be put across the fingerboard, passed down upon it and then lightly fixed to it by two set-screws at the two ends. The inner face of the clamp was lined with leather to imitate the ball of the fingers of the violinist and to prevent damage to the strings.

A few remarks are here necessary. In actual practice, when the violinist stops down the string so as to elicit a note of higher pitch, he generally takes the bow up rather nearer the bridge so as to preserve the relationship between the vibrating length of the string and the distance of the bow from the bridge. Strictly speaking, this should also have been done in the present investigation. But as it would have been somewhat troublesome and involved the risk of errors in the adjustment of the position of the bow, it was decided to keep the bow in a fixed position somewhat close to the bridge and to find the relationship between the bowing pressure and the pitch of the string under these conditions. We have already seen that when the bow is fairly close to the bridge,

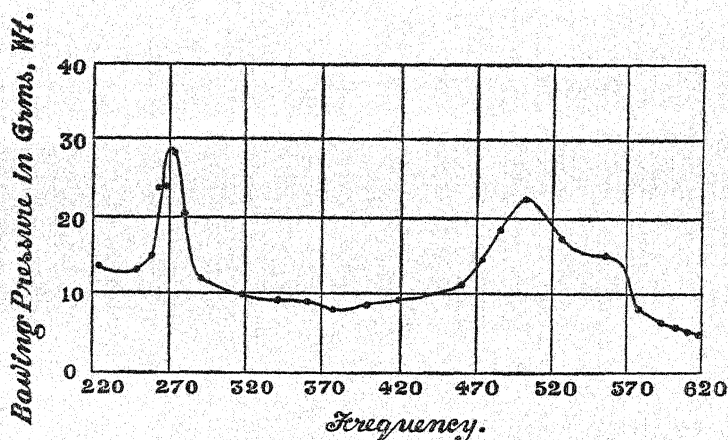


FIG. 3.—Relation between Bowing Pressure and Pitch (without Mute).

the bowing pressure necessary varies practically in inverse proportion to the square of the distance of the bow from the bridge. Accordingly, the effect of keeping the bow in a fixed position when the pitch is altered, instead of it bringing it nearer the bridge at each stage, is to decrease the bowing pressure necessary in a progressive and calculable ratio. This effect does not accordingly interfere with our observation of the characteristic changes of bowing pressure with pitch, which are connected with the changes in the forced vibration of the bridge and belly of the violin brought about by the change in the frequency of excitation.

The graph in Fig. 3 represents the relationship between bowing pressure and pitch within a part of the range of tone

of the violin which includes the first three of the natural frequencies of vibration of the body of the instrument. The particular violin used was of German make, marked copy of Antonius Straduarius, the bridge being of the usual Straduarius model. The experiments were made on the 4th or G-string, stopped down to various pitches. It will be noticed that the graph for the bowing pressure shows pronounced maxima and minima. There is a strong maximum at 270, another maximum between 470 and 520, and a distinct hump between 520 and 570. These maxima pretty nearly coincide in pitch with the first three maxima of intensity of the fundamental in the tone of the violin as estimated by ear, in other words with the frequencies of maximum resonance of the instrument to the gravest component of the force exerted on it by the vibrating string. The maximum lying between 470 and 520 is specially interesting as this region exhibits the well-known phenomenon of the 'wolf-note.' In the ascending part of this portion of the graph, and especially at and near the peak of the curve, it is found that when the pressure of the bow is less than the minimum required to elicit a steady tone with a well-sustained fundamental component, we get 'cyclical' or beating tones of the kind described and illustrated by me in previous papers.* The rapidity of the beats depends on the pitch of the tone which it is attempted to elicit, and also on the pressure and speed of the bow. A similar tendency to production of a 'wolf-note' though not so striking, is also manifested in the part of the graph between 520 and 570. The maximum in the region of 250 to 285 does not show a similar tendency, at any rate to any appreciable extent. It would appear that the gravest resonance of the violin chiefly involves a vigorous oscillation of the air within the belly of the instrument, but not so vigorous an oscillation of the bridge and belly as in the second and third natural modes of vibration which show the wolf-note phenomenon. Further evidence on this point is furnished by experiments on the effect of putting a load or mute on the bridge of the violin as will be referred to in the following section.

The formula for the bowing pressure quoted on page 25 enables the variation of bowing pressure shown in Fig. 3 to be explained. In the series

* Bulletin No. 15, also Phil. Mag. Oct. 1916.

$$\sum_{n=1}^{n=\infty} K_n B_n \frac{\sin \left(\frac{2n\pi t}{T} + e_n + e_n' \right)}{\sin \frac{n\pi x_0}{L}},$$

of which the graph practically determines the bowing pressure required, the B_n 's stand for the amplitudes of the different partial vibrations of the string, and the K_n 's are quantities which are practically proportional to the corresponding partial components of the forced vibration of the bridge transverse to the string at its extremity. In the case of a string bowed near the end,

$B_n / \sin \frac{n\pi x_0}{L}$ varies nearly as $1/n^3$ and thus decreases rapidly as n

increases. Further, in the part of the range of violin-tone covered by the graph in Fig. 3, the fundamental component of the vibration of the bridge should obviously be well marked, and K_1 would therefore be of the same order of quantities as K_2 , K_3 , etc., or even larger. Hence the value of the series given above would be principally determined by its leading term proportional to K_1 , and the variation of bowing pressure with pitch would practically follow the fluctuations of K_1 , in other words would follow the variations in the amplitude of the fundamental component in the forced vibrations of the bridge and pass through a series of maximum values at the successive frequencies of resonance of the instrument. This is practically what is shown by the experimental results for the bowing pressure appearing in Fig. 3, and the graph gives us an idea of the *sharpness* of the resonance of the instrument at each of the frequencies referred to. It must be remembered, however, that the bowing pressure required is also influenced by the terms proportional to K_2 , K_3 , etc., that is by the resonance of the instrument to the second, third and higher partial components of the vibration, and some evidence of this also appears in the graph in Fig. 3. For instance, though the first resonance of the instrument was actually found to be at a pitch of 284, the peak of the curve for the bowing pressure is at about 270 as can be seen from Fig. 3. This appears to be a consequence of the fact that the course of the curve is to some extent modified by the resonance of the *octave*. It is obvious that corresponding to the resonance of the instrument to the fundamental tone in the range of pitch from 470 to 570, the octave should be strongly reinforced when the pitch lies

within the range 235 to 285 ; hence the peak of the curve for the bowing pressure instead of being at 284 is actually shifted towards a lower frequency (270) as is seen from Fig. 3.

Obviously, the experiments described in this section may be extended in various directions. The curves for the three other strings of the violin, and especially over the whole of the possible range of pitch of the tone of the instrument, and the differences between the curves obtained with the different strings would deserve investigation. The differences between different violins could obviously be studied by this method, and the constants K_1 , K_2 , etc., for any particular violin and string and for various pitches may be found experimentally by study of the free and forced oscillations of the bridge, and used for a theoretical calculation and comparison with experiment of the bowing pressure required for exciting the tone of the instrument.

SECTION VI.—EFFECT OF MUTING ON BOWING PRESSURE.

Perhaps the best illustration of the close relation existing between the forces required to be exerted by the bow and the

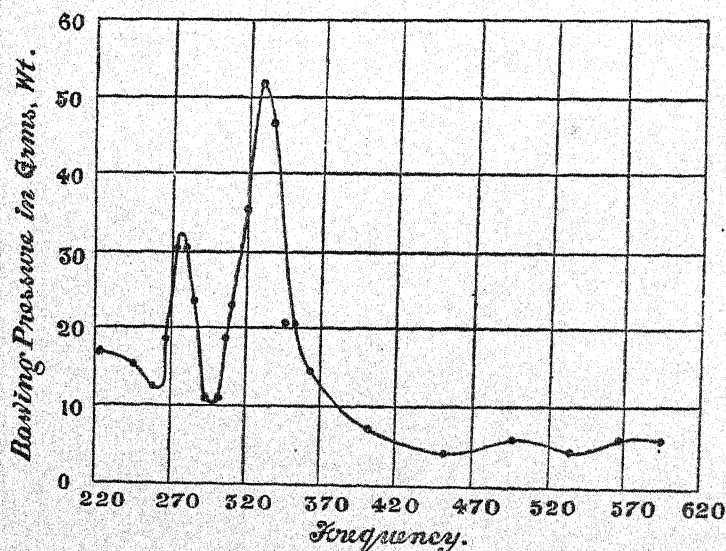


FIG. 4.—Relation between Bowing Pressure and Pitch (with Mute).

communication of vibrations from the string to the bridge and belly of the instrument and thence to the air, is furnished by the

effect of putting a mute on the bridge on the bowing pressure required for eliciting a musical tone. Very striking results on this point may be obtained with the aid of the mechanical player. Fig. 4 illustrates the relation between bowing pitch and bowing pressure observed experimentally, the G-string of the violin being used as in Fig. 3, and the results shown in Fig. 3 and Fig. 4 being obtained under the same conditions except that in the latter case, a brass mute weighing 12.4 grammes was clamped to the bridge, while in the former case, the bridge was unmuted. The great difference between the two cases is obvious, and the change in the form of the graph for the bowing pressure shows a very close analogy with the change in the character of the forced vibration of the instrument and the intensity of the tone of the violin over the whole range of pitch produced by application of the mute.

In view of the discussion of the graph for the bowing pressure contained in the preceding section, and the theoretical treatment of the effect of muting already given by me in previous papers* it is perhaps not necessary to enter here into a detailed examination of the subject, and it may suffice briefly to draw attention to some of the features appearing in the graph in Fig. 4. It will be noticed that there is a peak in the curve at a frequency of about 280. This is the pitch of the first resonance of the instrument, the fundamental component of the vibration being re-inforced, and in this case, the position of the peak in the curve for the bowing pressure is not appreciably influenced by the resonance of the octave as in Fig. 3. It is clear that the pitch of the first resonance of the instrument is hardly influenced at all by the application of a load of 12.4 grammes to the bridge, and the bowing pressure required at the first peak of the curve is nearly the same as in Fig. 3. The first natural mode of vibration of the violin does not therefore appear to involve any very large vibration of the bridge. Following the peak at 280, we have in fig. 4 a very high peak near 330 which is the pitch of the 'wolf-note' as lowered by the mute of 12.4 grammes. The great lowering of pitch (from 490 to 330) shows that the second mode of vibration of the body of the violin involves a very large vibration of the

* Phil. Mag. October 1916, Nature October 1917, Phil. Mag. June 1918, and Bulletin No. 15, pages 143 to 151.

bridge, and the enormous increase of bowing pressure at the peak of the curve is also noteworthy. This can no doubt be explained on dynamical principles as due to the very greatly increased amplitude of the forced vibration of the bridge due to the loading. At higher pitches, the bowing pressure necessary falls off very rapidly, though one or two minor maxima (due to the resonance of the instrument in its higher modes as altered by the loading) are also obtained. The tone of the instrument in the higher ranges of pitch when muted is extremely feeble.

Further investigations which are worthy of being carried out would be the study of the effect of gradually increasing the mass of the mute on the graph for the bowing pressure, and also of putting the mute at different places on the bridge. In view of what has been stated above, it is clear that we may expect the changes in the form of the graph to follow closely the changes in the pitch of resonance of the instrument in its various modes produced by the loading.

SECTION VII.—OTHER APPLICATIONS OF THE MECHANICAL PLAYER.

The investigations described in the preceding sections may be extended in various directions. Some indications have already been given on these points, and it will suffice here to suggest some of the other possible applications of the mechanical player. As the instrument affords a means of bowing the violin at precisely measurable speeds and pressures, it furnishes a means by which the intensity of violin-tone and its variations with pitch may be quantitatively determined and compared with the indications of mathematical theory. Various questions, such as for instance the effect of heavier or lighter stringing, the effect of varying the pressure, speed, and the width of the region of contact of the bow, and the position of the bowed region on the tone-quality of the violin may be quantitatively studied with a degree of accuracy that cannot be approached in manual playing with its undetermined conditions. Further, the study of the tone-intensity, of the bowing pressure curves, and of the vibration-curves of bridge and belly of the instrument under quantitative conditions made possible by mechanical playing may be expected speedily to clear up various structural problems relating to the construction of the

violin, e.g. the effect of the peculiar form of the Stradivarius bridge, the influence of its position, the function of the sound-post and base-bar, the shape of the air-holes, the thickness, curvature and shape of the elastic plates composing the violin, and the influence of various kinds of varnish. The dynamical specification of the constants determining the tone-quality of any violin over the whole range of pitch may be regarded as one of the aims towards which these investigations are directed.

SECTION VIII.—SYNOPSIS.

The paper describes the construction of a mechanical violin-player intended for study of the acoustics of the instrument, and some of the investigations in which it has been applied. The principal feature in the player which is worthy of notice is that the conditions obtaining in ordinary musical practice are imitated with all the fidelity possible in mechanical playing, and the results obtained with it may therefore be confidently regarded as applicable under the ordinary conditions of manual playing. The following is a summary of the results obtained in the four investigations described in the present paper: (1) *Effect of the position of the bowed region on the bowing pressure*: it is shown that provided the speed of the bow is not too small, the bowing pressure necessary within the ordinary musical range of bowing varies inversely as the *square* of the distance of the bow from the bridge. (2) *Relation between bowing speed and bowing pressure*: it is shown that for very small bowing speeds, the bowing pressure necessary tends to a finite minimum value, and the increase of bowing pressure with speed is at first rather slow, but later becomes more rapid. (3) *Variation of bowing pressure with pitch*: the graph for the bowing pressure for different frequencies shows a series of maxima which approximately coincide in position with the frequencies of resonance of the instrument. (4) *Effect of muting on bowing pressure*: it is found that the mute produces profound alterations in the form of the graph. The bowing pressure necessary is increased in the lower parts of the scale and decreased in the higher parts of the scale. The peaks in the graph shift towards the lower frequencies in consequence of the alteration in the natural frequencies of resonance of the violin produced by the loading, and the change in the form of the graph is closely

analogous to the change of the intensity of the fundamental tone of the instrument produced by the muting.

Some further possible applications of the mechanical player are also indicated in the paper.

III. Mechanical Illustration of the Theory of Large Oscillations and of Combinational Tones.

By Bhabo Nath Banerji, M.Sc., Assistant Professor of Physics
in the Calcutta University.

(Plate III.)

CONTENTS.

- SECTION I.—Introduction.
- SECTION II.—Description of Apparatus.
- SECTION III.—Single Forcing.
- SECTION IV.—Double Forcing.
- SECTION V.—Synopsis.

SECTION I.—INTRODUCTION.

It is well known that the principle of superposition of small motions is valid only for systems with infinitesimal amplitudes of vibration, and that the cases in which it fails owing to the finiteness of the amplitudes may often rise to considerable importance in acoustics. Referring to the principle of superposition Helmholtz remarks: "There are certain phenomena which result from the fact that this law does not hold with perfect exactness for vibrations of elastic bodies which though almost always very small are far from being infinitesimally small. For infinitesimally small motions, the moving forces excited by the mutual displacement of the particles of the oscillating medium are simply proportional to these displacements, but as soon as these vibrations become large the higher powers of the displacements begin to

sensibly influence the motion." On this basis, Helmholtz has founded an explanation of the presence in certain cases of harmonics in the tones emitted by simple vibrators and of the production of combinational tones within the human ear.

The experimental illustration of Helmholtz's theory is a matter of great interest, and has attracted the attention of many investigators. Amongst the most recent work on the subject may be specially mentioned, the contributions of E. Waetzmänn and W. Moser.* The production of combinational vibrations of strings by the simultaneous action of two simple harmonic forces varying the tension has also been described and illustrated by Professor C. V. Raman.† The present work taken up at the suggestion of Professor C. V. Raman proceeds on entirely different lines and is intended to furnish simple experimental and visible illustrations of the theory of large oscillations and of the production of combinational tones.

SECTION II.—DESCRIPTION OF APPARATUS.

The apparatus used was originally designed with an entirely different object, but has proved very well suited for the present purpose. It was at first intended to be used as an inharmonic synthesiser. The construction will be clear from Figs. (1) and (2) in Plate III. A wooden swinging arm is pivoted round an axle carrying ball-bearings so as to minimise friction (a bicycle-hub serves the purpose admirably). The axle is vertically fixed to a solid table, so that the swinging arm is free to move in a horizontal plane. (See Fig. 1 in Plate III.) The motion of the arm is checked and controlled by two strong steel springs tightly connected to it on either side, the other two ends of the springs being connected (subject to adjustment of tension) with two supports fixed to the table. The swinging arm carries a number of hooks from which a row of pendulums can be suspended (each by a pair of strings from consecutive hooks). When the pendulums are allowed to oscillate, they singly or jointly force a small oscillation of the

* Science Abstracts, Sec. A, May 1919.

† Bulletin of the Association, No. 11, 1914, and Physical Review, January, 1915.

Fig. 1

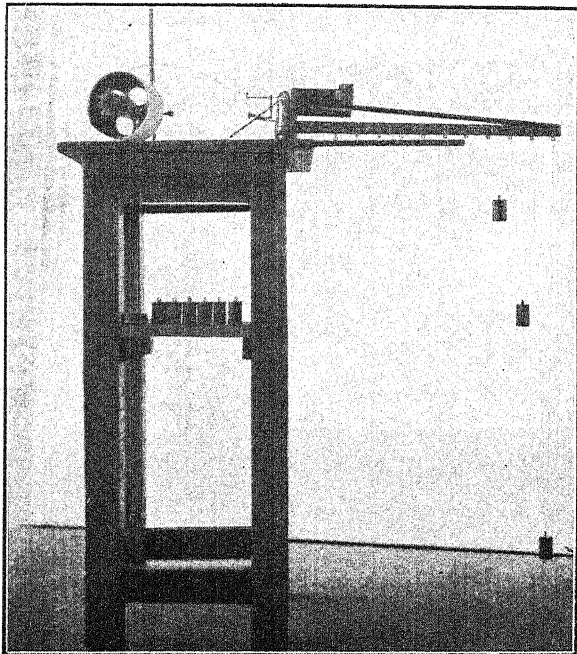
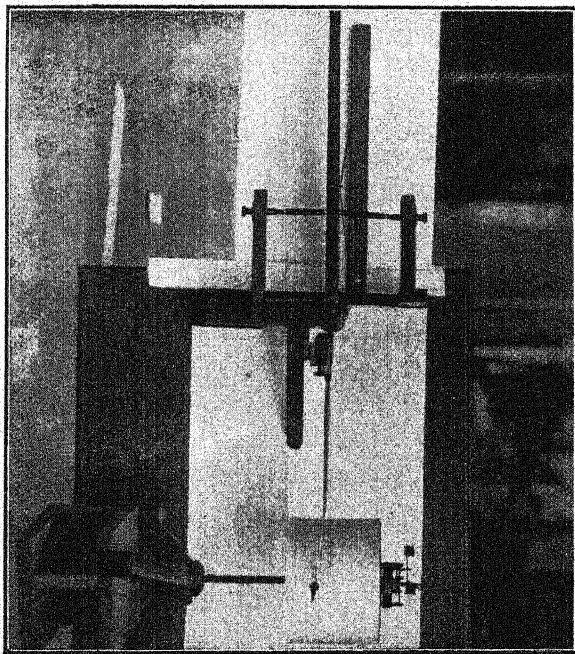
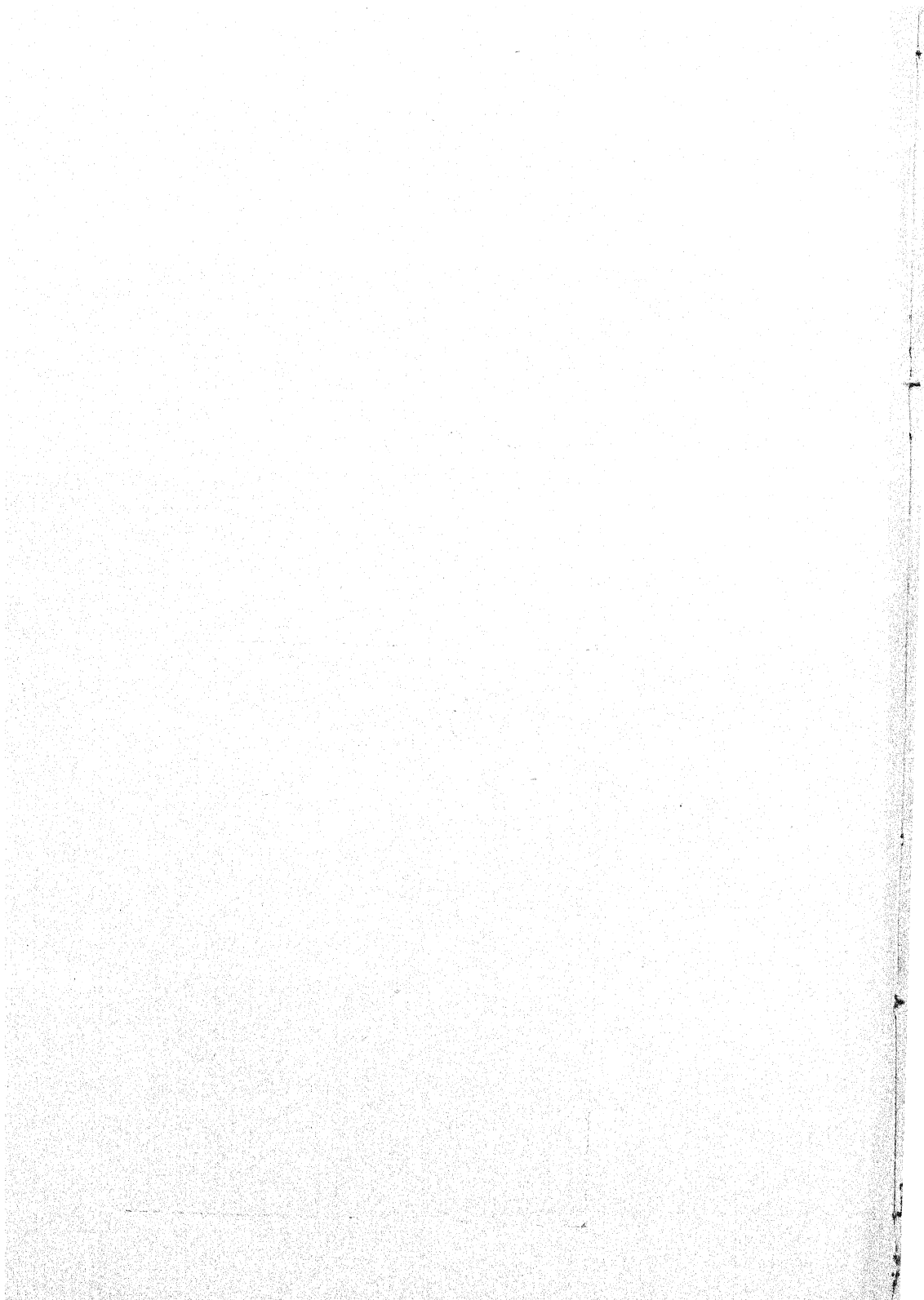


Fig. 2



Apparatus for Illustration of the Theory of Large Oscillations and Combinational Tones.



swinging arm. An arrangement is provided by which the oscillation of the arm may be magnified and recorded on paper if so desired. (This is not an essential part of the apparatus and may be left disconnected unless the records are required.) For this purpose the arm carries a vertical projecting pin which by working in a slot carried at the end of a long aluminium pen controls its movements. (See Fig. 2 in Plate III.) This pen is pivoted vertically at a small distance from the pin, the distance between pin and pivot being capable of adjustment with the aid of a screw. The pen carries a small reservoir of ink at the end—the flow of ink from the nib-point being made continuous by connecting the reservoir to the nib-point by a piece of cotton. This nib-point rests on a drum which can revolve about a horizontal axis by a clockwork arrangement and is covered by a sheet of paper.

The swinging arm in the apparatus as described above will be referred to as the symmetric vibrator in what follows—symmetric because the force of restitution for any displacement on either side of the position of equilibrium is the same. The swinging arm as controlled by the spring has a natural frequency of its own, its vibration when started being rather rapidly damped out on account of friction in the ball-bearings.

In order to make the vibrator asymmetric when desired, the following mode of attachment was resorted to. In addition to the two steel springs, two steel wires meeting at a very obtuse angle control the movement of the swinging arm. (See Fig. 2 in Plate III.) This attachment causes the force of restitution acting on the arm to be unsymmetrical. The wires tend strongly to resist stretching and consequently the restoring force on one side of the equilibrium position for the same displacement is not the same as that on the other side.

The vibrator in either case (symmetric or asymmetric) has a natural frequency determined by its inertia and the strength of the springs. A pendulum hanging from the vibrator when set in periodic oscillation exerts a periodic force on the latter—the magnitude of the force depending on the weight of the pendulum bob and its maximum angular displacement, and the period of the force is the same as the period of oscillation of the pendulum. If two or more simple harmonic forces of different magnitudes and periods be

required to be superposed on the vibrator, the same number of pendulums (which may be called the forcing pendulums) should be used, the magnitude being adjusted by the weight of the bob and the period by the length of suspension. Moreover if two forces at different phases are to be superposed on the vibrator, this can be done by timing the starting of the two forcing pendulums to correspond to the required phase difference. The swinging arm in all cases will vibrate under the joint action of all the forces acting simultaneously upon it. This mode of vibration of the vibrator can be recorded by the movement of the inking pen on the horizontally rotating drum covered with white paper, the ink traces showing the time-displacement curve of the vibrator.

From the vibrator an exploring pendulum can be suspended and will then act as a resonator, for if in the resultant oscillation of the vibrator a simple harmonic component synchronous with the periodic time of the exploring pendulum be present, the latter will be set in vigorous oscillation. The sharpness of this resonance will depend on damping and hence the exploring pendulum should be fairly heavy. This pendulum while showing its resonance effect reacts on the swinging arm being connected to the latter in the same way as the forcing pendulums. This mode of vibration of the swinging arm in the presence of the resonators will be illustrated in the course of the paper.

It will be noticed that the pendulum method of illustration bears an analogy to the recent work of Professor E. H. Barton and Miss Browning on forced oscillations described in the "Philosophical Magazine" (1918 and 1919).

SECTION III.—SINGLE FORCING.

I started a forcing pendulum with a small amplitude and recorded the ink trace of the vibrator.

FIG. (3). 

FIG. (4). 

FIG. (3) represents the vibration of the symmetric vibrator when a simple harmonic force of small intensity acts on it. The force is small, for the angular motion of the forcing pendulum on

which the former depends is very small. Here the curve is a simple sine curve; the motion being small, the first power of the displacement determines it.

FIG. (4) represents the vibration of the asymmetric vibrator under the action of simple harmonic force of small intensity. The asymmetry in the vibration is shown by the long flat portion of the curve which corresponds to very small movement of the vibrator against the wire attachment on one side, whereas on the other side the amplitude is greater though the curve is nearly flat in consequence of the higher powers of displacement determining the motion.

I then started the forcing pendulum with a large amplitude and obtained the following ink traces of the vibrator.

FIG. (5). 

FIG. (6). 

FIG. (5) represents the large oscillation of the pendulum as perfectly traced by the symmetric vibrator. The curve is not a simple sine curve and unlike fig. (3) the crests and troughs are flattened out, the remaining portions becoming comparatively steeper. The shape of the curve indicates the complexity of the vibration caused by the presence of the higher harmonic components in the resultant vibration, and the perfect symmetry on both sides indicates the absence of the even harmonics.

FIG. (6) represents the vibration of the asymmetric vibrator under the large amplitude of the forcing pendulum. It is very nearly the same as fig. (4) except that the asymmetry is more marked, and the nearly flat tops of the former curves have become flatter still, this portion being traced in the same circumstances as fig. (5).

For an analysis of the above vibrations, I started the forcing pendulum with a considerable amplitude and determined its period by means of a stop watch. I then made the length of the exploring pendulum to start with a little longer than that of the forcing pendulum and observed the effect of the latter on the former for all lengths right up to the shortest possible.

With the symmetrical vibrator it is found that the lengths of the exploring pendulum for which there is maximum resonance correspond with periodic times which are respectively the same and as one-third the period of the forcing pendulum. For all intermediate lengths the exploring pendulum takes up no regular oscillation except for lengths which are nearly equal to the resonating lengths. In the latter cases it is observed that the exploring pendulum takes up a periodic oscillation with varying amplitudes corresponding to beats, the frequency of which diminishes as the resonating length is approached. This phenomenon of beats helps the final adjustment of pendulum length corresponding to a resonance frequency. The periodic time for the latter being determined after every such adjustment confirms the presence of the fundamental as well as the third harmonic. It is observed that the length of the exploring pendulum resonating to the fundamental is longer than the forcing pendulum. The explanation of this lies in the fact that the periodic time of the pendulum with a large amplitude corresponds to a slightly longer pendulum with a smaller amplitude. The angle of swing of the resonant pendulum is much greater for the third harmonic than for the fundamental. It is also observed that the exploring pendulum set to give the octave shows no resonance. When the forcing pendulum is started with a lesser amplitude there is a very considerable decrease in the intensity of resonance, the third harmonic dying out more rapidly than the fundamental.

The observations described above are fully explained when we consider the nature of the forces exerted on the vibrator by the swinging pendulum. For small oscillations, the pendulum exerts a simple harmonic force synchronous with its own oscillation on the swinging arm from which it is suspended. But when the oscillation is large, this reaction includes also a third harmonic as may be readily shown.

The differential equation of motion of pendulum is

$$Ml \frac{d^2\theta}{dt^2} = -Mg \sin \theta$$

If α be the semivertical angle of swing of the pendulum the first approximate solution gives

$$\theta = \alpha \cos wt, \text{ where } w = \sqrt{\frac{g}{l}}$$

the second approximate solution becomes

$$\theta = \alpha' \cos w't - \frac{\alpha'^3}{192} \cos 3 w't$$

where $w' = w \left(1 - \frac{a^2}{16}\right)$ and α' is slightly different from a .

The horizontal reaction on the vibrator which is

$$Mg \cos \theta \sin \theta \text{ or } Mg \left(\theta - \frac{2}{3} \theta^3 \right)$$

approximately becomes after substitution

$$Mg \left\{ \alpha' \cos w't - \frac{1}{6} \alpha'^3 \cos 3 w't \right\} \text{ approximately.}$$

If n be the natural frequency of the vibrator, its oscillation will be represented by

$$\ddot{y} + n^2 y = \alpha' \cos w't - \frac{1}{6} \alpha'^3 \cos 3 w't$$

the constant factors depending on Mg and the inertia of the vibrator being omitted from the coefficients of the force terms in the equation. The solution then becomes

$$y = \frac{\alpha'}{n^2 - w'^2} \cos w't - \frac{1}{6} \frac{\alpha'^3}{n^2 - 9w'^2} \cos 3 w't$$

It will be noticed from the above equations as well as by the mechanical analysis that the complexity of the vibration in fig. (5) is due to the presence of the third harmonic, the symmetry of the curve being explained by the absence of the even harmonics. The two resonances obtained are due to the corresponding forces exerted on the vibrator by the large oscillations of the forcing pendulum. The angular swing of the resonating pendulum corresponding to the third harmonic is large; for the forced motion of the vibrator having this frequency is proportional to the factor $\frac{1}{n^2 - 9w'^2}$, and is thus great, the natural frequency n of the vibrator being in the actual experiment nearly equal to $3w'$ the frequency of the resonant pendulum. Then again the intensities of resonance of the primary and the third harmonic depend respectively on α' and α'^3 which explains the phenomena that with small values of α' the third harmonic becomes very small, being dependent on α'^3 which

will then be negligibly small. It will be noticed that the explanation of the presence of the third harmonic in the motion given above does not involve any assumption that the springs controlling the motion of the vibrator deviate appreciably from Hooke's Law. Such a deviation, if it did exist however, would produce results practically analogous to those described above.

With the asymmetric vibrator it is found that the lengths of the exploring pendulum for which there is maximum resonance corresponds with periodic times which are (1) equal; (2) one-half; and (3) one-third of the period of the forcing pendulum. The angle of swing for the fundamental resonance is small, the length of the exploring pendulum being slightly greater than that of the forcing pendulum as remarked before. The second and third harmonics decrease in intensity when the amplitude of the forcing pendulum is decreased. In the case of the third harmonic the decrease is very marked, but the second harmonic persists even with diminished amplitudes. The presence of the third harmonic needs no further explanation in view of what has been stated above. In forming the approximate differential equation of motion for the vibrator for the present case, the force corresponding to the third harmonic may be omitted for simplicity. The equation then becomes

$$\ddot{y} + n^2 y + \beta y^3 = a \cos \omega t$$

where β is the constant of asymmetry.

A solution to the above is

$$y = k + \frac{a}{n^2 - \omega^2} \cos \omega t - \frac{\beta}{2(n^2 - 4\omega^2)} \left(\frac{a}{n^2 - \omega^2} \right)^2 \cos 2\omega t$$

The asymmetric factor β is involved in the amplitude coefficient of the second harmonic and hence the persistence of the latter though with smaller intensity for lesser forcing as represented by the presence of a^2 in the coefficient. That the intensities of resonance depend on the natural frequency of the vibrator is also evident from the solution.

The following curves illustrate the modification of the vibration of the vibrator in the presence of the resonator, the vibration curves without the resonators having been reproduced before. In getting the records, the forcing pendulum is started with a big amplitude and the exploring pendulum adjusted to give the particu-

lar resonance is allowed to hang undisturbed. This second pendulum takes up the oscillation by resonance and the records of the vibration are then obtained.

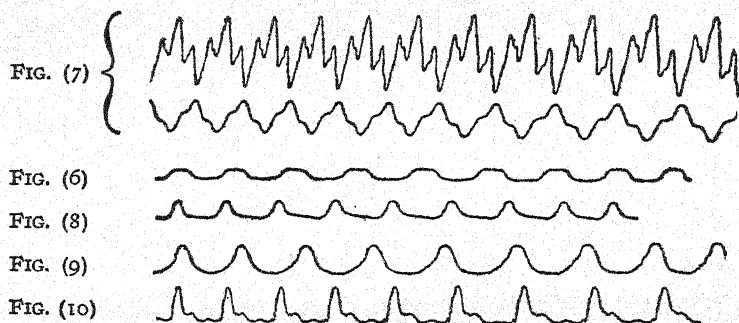


FIG. (7) represents the vibration of the vibrator in the presence of the resonator which corresponds to the third harmonic feebly present in the forcing pendulum. It will be noticed that the presence of the resonant pendulum results in an enormous magnification of the third harmonic in the resultant motion of the vibrator. FIGS. (8), (9) and (10) represent the vibrations of the asymmetric vibrator—fig. (8) is the vibration of the vibrator in the presence of a resonator which is the octave of the force acting; fig. (9) is in the presence of a resonator which corresponds to the third harmonic; fig. (10) is in the presence of resonators corresponding to the octave and the twelfth combined. It is evident from these figures that the effect of the resonant pendulums is very greatly to increase the amplitudes of the upper partials in the resultant vibration.

SECTION IV.—DOUBLE FORCING.

I next pass on to the case of two periodic forces acting on the vibrator simultaneously. For this purpose another pendulum is suspended from the vibrator just by the side of the first forcing pendulum. The two forcing pendulums are then vigorously started. With the symmetrical vibrator it is found that the exploring pendulum responds to primaries and twelfths of each of the forces acting. In fact, one or other of the forces acting singly gives all the resonances obtained, all observations being quite similar to those noticed before in the case of single

forcing. No resonance is observed when the exploring pendulum is adjusted to correspond to either the sum or the difference of the frequencies of the two forcings. The vibrator of course in the presence of the two forces takes up a vibration as if the algebraic sum of the two forces act singly on it.

A special case of the resultant vibration when the periods of the two primary forces acting are nearly equal is represented by the following ink trace of the vibrator. The phenomena of beats which the algebraic sum of the two components would give, the frequency of the beats being the difference of the frequencies of the two components, is clearly illustrated in Fig. (II).



FIG. (II).

Having next made the vibrator asymmetric I started the two forcing pendulums vigorously and obtained resonance of the exploring pendulum in the cases shown in the following table. The choice of the two forcing pendulums was such that the sum and difference of their frequencies were not the same as any of the higher harmonics of either of the forces.

TABLE.

Period and frequency of forcing pendulums.	Resonating pendulum.		Relation.	REMARKS.
	Period.	Frequency.		
$T_1 = 2.08$ or $p = .48$	2.08	.48	Primary.	Corresponding to p
	0.90	1.11	Primary.	„ to q
	1.04	.96	Octave.	„ to $2p$
$T_2 = 0.9$ or $q = 1.11$	0.45	2.22	Octave.	„ to $2q$
	0.69	1.44	Twelfth	„ to $3p$
	0.63	1.59	Summational.	„ to $p + q$
	1.60	.63	Differential.	„ to $p - q$

The differential equation of motion for the above case is represented as

$$\ddot{y} + n^2 y + \beta y^3 = f \cos pt + g \cos qt$$

where p and q are the frequencies of the forcing pendulums, f and g the intensities of the respective forces acting. In the above equation the small forces of frequency $3p$ and $3q$ present in the corresponding large amplitude forcing pendulum have been left out of the equation for simplicity, the present intention being the illustration of the theory of combinationals. Hence a solution to the above approximate equation will not contain the corresponding terms, though they are actually present.

The solution of the above equation is of the form

$$y = K + A \cos pt + B \cos qt + C \cos 2pt + D \cos 2qt \\ + E \cos \sigma t + F \cos \delta t$$

where $\sigma = p + q$ and $\delta = p - q$.

The results shown in the table are thus explained. The combinationals of the first order where σ represents the summational and δ the differential have for their intensity coefficients

$$E = \frac{\beta}{n^2 - \sigma^2} \cdot \frac{f}{n^2 - p^2} \cdot \frac{g}{n^2 - q^2} \\ F = \frac{\beta}{n^2 - \delta^2} \cdot \frac{f}{n^2 - p^2} \cdot \frac{g}{n^2 - q^2}.$$

The above equations explain conclusively the facts that are actually observed in the experiment. An increase in β the factor of asymmetry increases the combinationals. The alteration in the constant of asymmetry is mechanically illustrated by an alteration in the diameter or tension of the wires controlling the motion of the vibrator. An increase in f or g the magnitude of the primary forces acting increases the intensity of the combinationals. This is an illustration of the well-known fact that for the production of the combination tones the generating tones must be loud and well sustained. Then again the factor $n^2 - \sigma^2$ or $n^2 - \delta^2$ indicates that the proximity of the frequency of the tone to the natural frequency of the vibrator increases that tone considerably. With my apparatus the summational approaches the above condition and hence its comparatively stronger resonance than with the differential.

It was also found that the two forcing pendulums with the exploring pendulum resonating to a combinational formed a system of which any two when vigorously started produced resonant oscillation in the third. For if a and b be the frequencies of the forcing pendulums and c that corresponding to the summational $a+b=c$ and

$$c - a = b$$

$$c - b = a$$

so that if c and a or c and b are started the respective resonances will be a or b corresponding to the differentials as the equations will show.

The following curves show the mode of vibration of the vibrator in the asymmetric system.



FIG. (12).

FIG. (12) is the vibration curve of the vibrator under double forcing and in the absence of any resonator. It also illustrates the phenomena of asymmetric beats.

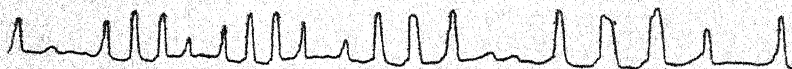


FIG. (13).

FIG. (13) represents the modified vibration curve of the vibrator in the presence of a resonator tuned to the summational. It will be noticed that the component motion corresponding to the summational frequency becomes markedly more prominent and obvious to inspection.

The asymmetric vibrator together with the resonators may roughly be taken as the mechanical representation of the human ear, the drum skin being compared to the asymmetric vibrator and the resonating chords of the basilar membrane to the exploring pendulums. From what has been said before, it will be quite clear that when two loud sources of sound affect the drum skin of the ear, the chords of the basilar membrane corresponding to the

harmonics and combinational tones are excited and the corresponding sensations are produced.

The vibration of the drum skin is not the simple sum of the motion due to the two primary disturbances acting separately but considerably modified because of its asymmetric configuration and also its connection to the inner vibrating parts of the ear including the basilar membrane. We have seen in the foregoing that the motion of the asymmetric vibrator under the joint action of two periodic disturbances may be greatly modified by the presence in connection with it of resonators tuned to the frequency of the combinational vibration, the effect being to magnify the amplitude of this part of the disturbance. The experiments thus also illustrate Helmholtz's remark that even where the production of the combinational tones occurs within the ear, the association of the ear with a suitable resonant cavity connected with it may tend to reinforce the combinational tones. Further, it is also not impossible that a similar reinforcement of the combinationals may occur within the ear itself in consequence of the connection with the drum skin of the internal parts capable of vibration and having free periods of their own. The natural frequency of the drum skin unlike the mechanical analogue is small in comparison with the frequency of the audible disturbances affecting it, and hence the stronger production of the differentials than the summationals in the human ear unlike the mechanical analogue where the summational is the stronger.

SECTION V.—SYNOPSIS.

The paper gives a short account of experiments illustrating the production of harmonics and combinational tones in systems having finite amplitudes of vibration. The apparatus consists of a swinging arm pivoted round a vertical axle and free to oscillate in a horizontal plane; the forces controlling this oscillation may be made either symmetric or asymmetric as desired. From the swinging arm a number of pendulums are suspended which can be set in oscillation. The behaviour of the pendulums can be observed, and provision is made by which the motion of the swinging arm due to their reactions may be recorded if desired, in the form of a time-displacement graph.

The apparatus has been used to demonstrate the following :—

(1) The presence of the third harmonic in the oscillations forced by a pendulum swinging through a large amplitude.

(2) The presence of the second harmonic in the oscillation of an asymmetric vibrator subject to a simple-harmonically varying force.

(3) The influence of a resonator of the appropriate frequency attached to the vibrator in either case in magnifying those components of its motion.

(4) The production of beats under double forcing of symmetric and asymmetric systems.

(5) The production of combinational oscillations of an asymmetric vibrator under double forcing, and the influence of the free period of the oscillator on the magnitude of the components of combinational frequencies in its motion.

(6) The influence of resonators tuned to the combinational frequencies upon the motion of the asymmetric system when connected with it.

In conclusion, the author wishes to express his cordial thanks to Prof. C. V. Raman for the facilities put at his disposal and for constant interest and encouragement in his research.

IV. Some Phenomena of Laminar Diffraction observed with Mica.

By Phanindra Nath Ghosh, M.A., Lecturer on Optics in the
University of Calcutta.

(Plate IV.)

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SECTION II.—Micro-structure of the Striae in Mica.
SECTION III.—Spectrum of the Laminary Diffraction Pattern.
SECTION IV.—Effects observed close to the Striae.
SECTION V.—Intensity, Colour and Polarisation of the Large-Angle Diffraction.
SECTION VI.—Mathematical Theory of the Phenomena.
SECTION VII.—Synopsis.

SECTION I:—INTRODUCTION.

In a paper recently contributed to the Proceedings of the Royal Society,* the author has described and explained the interesting phenomena observed when a sheet of mica is examined by the Foucault test or in the Toepler "Schlieren" apparatus as it is otherwise called. Certain lines or "striae" on the surface of the mica appear luminous and beautifully coloured, the colour depending on the angle at which the mica is held to the light incident on it in the apparatus. It was shown that the striae are the boundaries between regions of the mica having slightly different thicknesses, and it was pointed out in the paper that the colour of a stria as observed in the Foucault test is complementary

* "On the Colours of the Striae in Mica." Proc. Roy. Soc., 1919, Vol. 96, pp. 257-266.

to the colour of the central fringe in the laminary diffraction-pattern produced by it and observed in rear of the sheet of mica when plane waves are allowed to traverse it. Attempts were made to reproduce the colours observed in the Foucault test by using glass plates with artificially prepared laminar boundaries, obtained by etching out a very thin layer over part of the surface of the glass with dilute hydrofluoric acid. These attempts were not very successful, and this failure was ascribed in the paper to a want of sufficient abruptness or sharpness in the diffracting edges thus prepared. On the ordinary elementary theory, a sharp laminar boundary should give a diffraction-pattern which is more or less exactly symmetrical in configuration about the central fringe.* In practice, etched glass plates give diffraction-patterns which are markedly asymmetrical,† the fringes on one side of the centre being much brighter than those on the other. Careful observations showed that even in the case of mica, the laminary diffraction fringes produced by the striae often showed distinct asymmetry though to a much less extent than in the case of etched glass plates. The central fringe which is strongly coloured often showed distinctly different tints at its two edges and was occasionally even completely bifurcated in colour. It was thought that a closer examination of these phenomena, and of the nature of the laminar boundaries in mica, would be of interest. The results of the investigation are presented in this paper.

SECTION II.—MICRO-STRUCTURE OF THE STRIAE.

The striae appear to the naked eye as fine hair-like lines on the surface of the mica when the latter is examined in diffuse light. Under the microscope, however, an interesting structure is revealed and the striae appear resolved into minute echelons, or staircases, the number of steps in the echelon being often considerable. Under direct illumination and moderate powers, the boundaries of the successive steps of the echelon appear in the bright field as fine dark lines which when the highest powers of the

* R. W. Wood, "Physical Optics," 1914 Edition, page 250. The symmetry should be exact when the phase-difference on the two sides of the boundary is π or any multiple of π . In other intermediate cases, the pattern is asymmetrical, but in a relatively minor degree.

† See, for instance, the photograph by Wood published in his book, *loc. cit.*

microscope are used may either remain visible as such or appear still further resolved into two, three, or more fine dark lines. Under oblique illumination, we have the opposite effect, the entire field of the microscope being dark and the successive edges of the echelon appearing as bright lines. It should be noticed that the successive edges in a stria do not all appear equally dark when it is seen under direct illumination, nor do they all appear equally bright when seen under oblique illumination. It is thus clear that the successive steps of the echelon generally represent unequal changes of thickness in the mica. Further, the steps are also generally of unequal width, and the width of a step may even vary from point to point along the length of a stria. In some cases a single boundary may be seen split up along its length into two or even into three boundaries. The width of a stria is a variable magnitude. Striae have been observed of which the entire width does not exceed $1/400$ mm. or even $1/600$ mm., i.e., about five wave lengths of sodium light, and broad striae have been observed which extend over $1/20$ th or $1/15$ th part of a mm., that is, from one hundred to one hundred and fifty times this wave-length. The number of steps is in some cases fifteen or twenty but is usually found to be between six and ten, and may occasionally be as small as one, two, or three.

Taking the case of a typical stria whose micro-photograph is shown in fig. 1 in the Plate, we find it has seven steps, the total width of the steps being about $1/100$ mm., i.e. about 20 wave-lengths.

The widths of the successive steps in this stria as actually measured with a micrometer were found to be as shown below:—

1st Position.	2nd Position.	3rd Position.
1. '0010 mm.	'0010 mm.	'0009 mm.
2. '0025 „	'0022 „	'0023 „
3. '0010 „	'0010 „	'0011 „
4. '0015 „	'0017 „	'0018 „
5. '0028 „	'0024 „	'0028 „
6. '0022 „	'0026 „	'0010 „

With the Jamin interferometer it was found that the optical retardation produced by the mica on the two sides of this stria differed by just half a wave-length of sodium light. On the assumption that this retardation is equally distributed between the

successive steps, the difference of retardation at each step is of the order of $1/14$ th of the wave-length of the D lines. But the actual thicknesses are of different magnitudes as is evident from the appearance of the different boundaries as seen under the microscope, and some of the steps probably therefore represent optical retardations even less than $1/20$ or $1/25$ th part of the wave-length of sodium light. It is remarkable that laminar boundaries representing such small optical retardations in a transparent plate are clearly seen as fine dark lines under direct illumination in the microscope. The visibility of the laminar edges is evidently closely connected with their power of diffracting light, and presents some interesting points of comparison with the question of the visibility of particles of much smaller dimensions than the wave-length of light in the microscope. It would seem worthwhile on a future occasion to examine the matter further and determine by direct observation the smallest thicknesses of the laminar boundaries which can be obtained in mica and which can be detected in the microscope under direct or indirect illumination.

The appearance of the laminar edges in etched plates of glass under the microscope forms a striking contrast with the phenomena observed in mica. They appear as blurred and ill-defined bands even under low powers, and become altogether indistinguishable when objectives of higher powers are used.

SECTION III.—SPECTRUM OF THE LAMINARY DIFFRACTION PATTERN.

As remarked in the introduction, the distribution of intensity and colours in the diffraction-patterns produced by the striae in mica often shows a distinctly noticeable asymmetry of which the explanation is evidently connected with the echelon-like nature of the boundary. A delicate method of observation which would render this asymmetry strikingly evident and susceptible of measurement became necessary and was ultimately found in the spectroscopic examination of the diffraction-pattern. When the laminar diffraction fringes are allowed to fall crosswise on the slit of a spectroscope, the spectrum as seen is found to be crossed by dark bands showing the colours obscured by interference in the different parts of the pattern. If the distribution of intensity in the laminar diffraction-pattern had been strictly symmetrical, we

Fig. 1

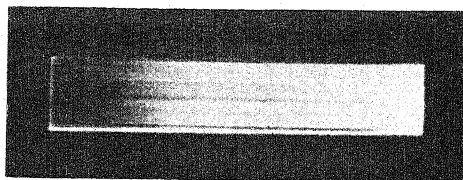
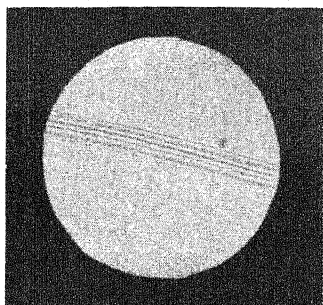


Fig. 2

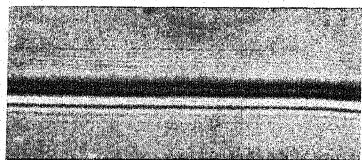
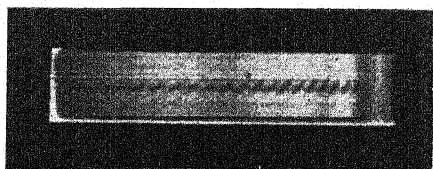


Fig. 3

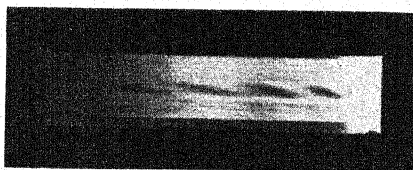
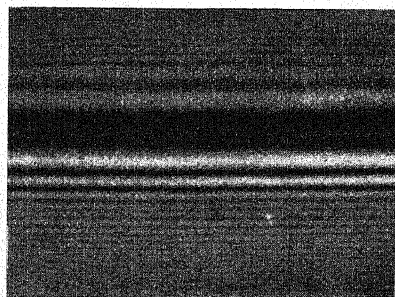
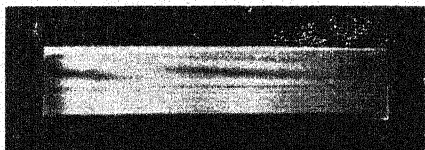
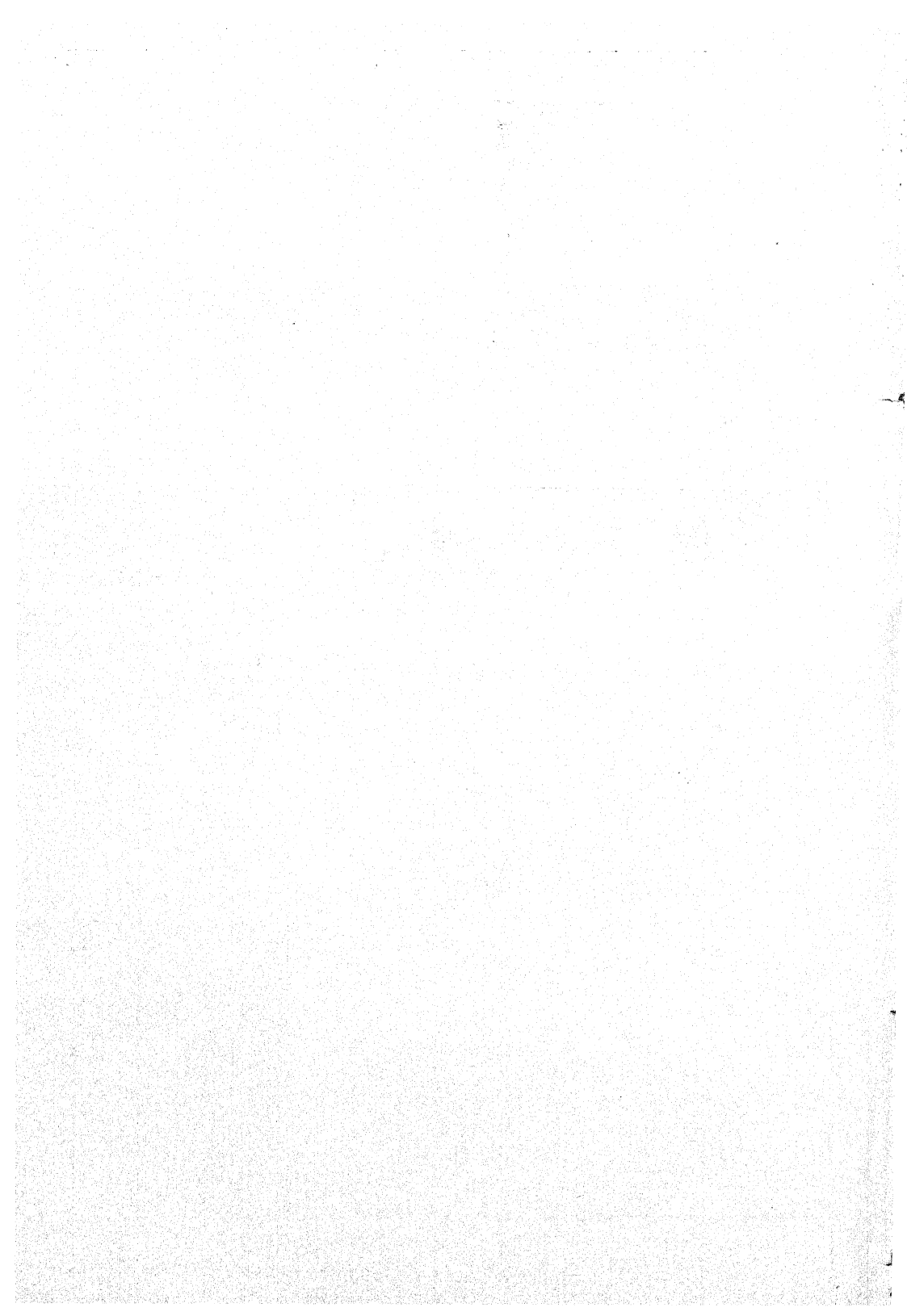


Fig. 4





should evidently see in the spectrum a system of bands with a configuration symmetrical about the central line. Actually, however, we obtain the very curious appearance of bands running obliquely through the spectrum (Figs. 2, 3, 4 in Plate IV).^{*} Fig. 2 which shows quite a large number of bands was obtained with a rather thick stria which showed merely a grey colour in the central fringe. Here every one of the bands is inclined, the inclination gradually diminishing as one proceeds towards the violet end of the spectrum. The actual shape of the individual bands running through the spectrum is an oblique \angle of which the tips are considerably fainter than the middle. The same characteristics are also shown by fig. 3 in which we find four oblique bands distributed practically equally throughout the spectrum, and the distribution of intensity above and below these bands in the spectrum is obviously quite different. The phenomena are still more clearly seen in fig. 4 which was obtained with a stria which gave a green-coloured central fringe. The obliquity of the two dark bands running through the spectrum and the asymmetric distribution of intensity above and below these bands is particularly noticeable in this figure. Fig. 5 represents the spectrum of the fringes due to a stria giving a half wave-length change in optical retardation. Here, though the band runs straight through the spectrum, a distinct obliquity in its position and an asymmetric distribution of intensity on the two sides of it were both noticed. The intensity was greater on one side of the fringe than on the other, and the fainter bands which accompany the central dark band on either side are much clearer and further from it on one side than on the other.

The relative optical retardation on the two sides of a stria should obviously be increased by tilting the mica, and it is interesting to watch the effect of this on the position and number of the bands in the spectrum of the laminary diffraction-pattern. The observations are best made with a stria which shows a small number of bands in the spectrum. It is found that as the mica is

^{*} It is found that superposed upon these bands, there also appear in the spectrum a set of very fine, numerous and rather diffuse bands running parallel to the slit of the spectroscope. These are the well-known bands due to the interference of the light transmitted through the mica with the light which has passed into the spectroscope after two or more internal reflections within the mica, and need no further remarks.

turned gradually from the normal to the oblique position with respect to the light incident on it, the bands shift towards the red, and the number of the bands visible in the spectrum also increases.

Phenomena which are roughly analogous to the above but not so beautifully clear and regular are obtained when the spectrum of the laminar diffraction of a very thinly etched glass plate is observed. The appearance of asymmetry in this case is considerably more exaggerated, the fringes on one side of the pattern being numerous and showing well-marked contrasts, and the fringes on the other side being few in number and hazy in outline.

SECTION IV.—EFFECTS OBSERVED CLOSE TO THE STRIAE.

The phenomena described in the preceding section are those observed at comparatively large distances (of the order of a meter) from the mica. It is of interest to examine the effects observed near the sheet of mica, say within a few centimeters of it, as we may naturally expect that the asymmetry of the pattern due to the finite width of the striae would be much more marked in their immediate neighbourhood. For the purpose of this study, it is found convenient to use striae which run more or less straight. The source of light is a slit placed at a sufficient distance from the mica and parallel to the direction of the striae, and illuminated by sunlight or the light of an electric arc. The diffraction-fringes formed in the rear of the striae may be observed through a microscope. For spectroscopic analysis of the pattern, the eyepiece of the microscope may be removed and the image of the diffraction-pattern formed by the objective may be allowed to fall on the slit of a direct-vision spectroscope or of a constant-deviation wavelength spectrometer. By racking out the objective of the microscope, the complete succession of phenomena commencing from the plane of the mica itself right up to any desired distance from it may be observed in quick succession, and a vivid idea obtained of the whole series of effects. The observations made in this manner, especially with the aid of the spectroscope as described above, are extremely useful in getting at a clear understanding of the whole case.

The phenomena noticed naturally depend a good deal on the particular stria under study, especially on the number, width, and height of the steps in the staircase and the total relative optical

retardation on the two sides of it. Nevertheless, certain general features are observed which are common to most of the cases studied. When the focal plane of the objective coincides as nearly as possible with the mica, the structure of the stria is clearly seen, the successive edges of the echelon appearing as fine dark lines. Even at this position, however, a number of very fine (practically equidistant) fringes may be seen bordering these edges, these being more marked on one side of each edge than on the other. The simplest case is that in which the echelon consists of but a single step. Occasionally striae may be obtained consisting of but a single edge not resolvable by a $\frac{1}{8}$ th inch Reichert objective. Even in such cases, however, the laminar diffraction-pattern at close quarters often shows a distinct and sometimes quite marked asymmetry, the fringes on one side being clearer, brighter and more numerous than on the other, and the central fringe appearing much darker than it is at a distance from the mica. These features persist till the focal plane is drawn away from the mica to a distance of a centimeter or two. Apart from these special characters, however, the pattern due to an echelon of a single step is very similar to what we should expect on the elementary theory in the case of a perfectly abrupt laminar boundary.

With striae consisting of two, three or more steps in the staircase, the effects are naturally more complex than in the case of a single step. As the focal plane is drawn away from the mica, the fringes bordering the successive edges rapidly broaden out and become superposed on each other. Soon, all trace of the structure of the stria is lost owing to this superposition, and the field of view may be then seen clearly differentiated into three parts the effects observed in which may be separately considered.

First, the central part of the field which is much darker than the rest of the field. The width of this part of the field differs for different striae, being large for the broad striae, and small for the narrow ones. This central region is seen filled with a succession of faint but markedly coloured fringes which are at first narrow and numerous, but widen out and become fewer in number as the focal plane is drawn away from the mica, till finally only one or two coloured bands remain in the middle of the field.

Secondly, the part of the field lying on the thicker side of the stria. This is the brightest part of the field, and contains numer-

ous well-marked fringes which at first are nearly equally spaced and extend to a considerable distance from the central part of the field. As the focal plane is drawn away from the mica, these fringes become broader, less bright, and their spacing becomes more unequal. Fewer fringes are also then visible.

Thirdly, the part of the field on the thinner side. This usually contains only a few faint and hazy diffraction fringes. The contrast between the second and third parts of the field is very marked but decreases gradually as we recede from the mica.

Corresponding to the foregoing changes in the microscopic appearance of the fringes, the spectrum of the diffraction-pattern also alters. In the central part of the field, a number of narrow *horizontal* bands is at first seen in the spectrum. As the objective recedes from the mica, these bands widen out, those on the thicker side of the mica slide away towards the red end of the spectrum, those on the thinner side slide away towards the blue end, and the remaining bands gradually assume an oblique position and tend to set themselves in the spectrum less inclined to the slit of the spectroscope as we recede from the mica. When the focal plane is within a centimeter or two of the mica, a large number of very dark and bright fringes may be seen in the region of the spectrum on the thicker side of the stria, and relatively few and more hazy fringes on the thinner side. These fringes in the spectrum gradually widen out and become less numerous and more hazy in outline as we recede from the mica.

Figs. 6 and 7 in the Plate illustrate the preceding remarks and represent microphotographs (in the light of the electric arc) of the diffraction-effects observed close to the striae, fig. 6 having been secured with the focal plane nearer the mica, and fig. 7 with the focal plane somewhat further away. As the light used was not monochromatic, the photographs do not convey an adequate idea of the very large number of fringes visible when the pattern is analysed by the spectroscope.

The diffraction-effects due to the laminar boundaries in etched glass plates observed in their neighbourhood through a microscope are not so striking as those observed with mica, the number of fringes visible, specially in the spectral analysis of the pattern, being much smaller than with mica. Apart from this, however, the effects observed in the two cases are broadly analogous,

the asymmetry being considerably more exaggerated in the case of etched glass plates.

SECTION V.—INTENSITY, COLOUR AND POLARIZATION OF THE LARGE-ANGLE DIFFRACTION.

On account of the extremely fine structure of the laminar edges forming the striae in mica, they possess the property of scattering or diffracting light in directions making large angles (up to 180°) with the light transmitted through or reflected from the plane surface of the mica. The feature of this large-angle diffraction which is most noteworthy is that the striae as seen by the light diffracted by them appear very *much more intensely* luminous on the *retarded* side of the wave-front and relatively quite feebly luminous on the other side. Observation also shows that when white unpolarized light is incident on the striae, the light scattered by them is often strongly coloured and also polarized. The investigation of these effects is a matter of some complexity, as they are found to depend on a number of factors, (a) the fine structure of the laminar edge and the relative optical retardation on the two sides of it, (b) the angle of incidence of light, and (c) the angle of diffraction. Further, though the problem of the large-angle diffraction of light by the edge of a semi-infinite *perfectly reflecting* screen has been investigated by Sommerfeld and shown to involve polarization effects, the corresponding problem of the large-angle diffraction by the edge of a thin lamina of transparent solid yet remains unsolved, and no theoretical guidance for research in this direction is therefore available. It is hoped to investigate these effects in detail experimentally when a suitable opportunity occurs.

SECTION VI.—MATHEMATICAL THEORY.

The observations considered in the preceding sections which require explanation are the following: (a) the appearance and visibility of the laminar boundaries in the microscope when focussed upon them in direct illumination; (b) the diffraction phenomena observed close to the striae, and (c) the effects observed at a distance, especially the obliquity of the bands appearing in the spectrum of the diffraction-pattern. The investigation of (a) is a question relating to the theory of microscopic vision

which the author hopes to be able to deal with in a later paper. The investigation of (b) and (c) strictly speaking requires an exact knowledge of the structure of the particular stria under observation. As our present purpose is, however, merely to get a general idea of the explanation of these effects, it is sufficient if the stria is assumed to be an approximately wedge-shaped boundary separating regions of the mica having slightly different thicknesses. Such a wedge-shaped boundary would diffract light in roughly (though not by any means in exactly) the same way as the equivalent staircase structure.

Considering now the effects in the neighbourhood of the

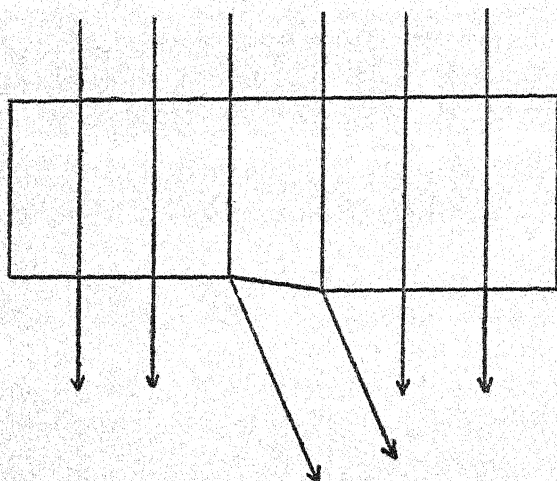


FIG. 8.

stria, fig. 8 showing the geometrical path of the rays gives us at once a general idea of the phenomena to be expected. The central relatively dark part of the field, the increased brightness and the numerous interference fringes seen in the region on the thicker side of the mica, and the relatively few and hazy diffraction fringes seen on the thinner side of the mica are all exactly what we should expect on the principles of the wave-theory. The following is the detailed mathematical treatment :—

The form of the wave front on emergence from the plate shown in fig. 8 would be that shown by the upper line in fig. 9. The effect at any point O in the field may be readily found in terms of Fresnel's

integrals. In fig. 9, r_0 is the distance of the pole from the point of observation and x_1, x_2 the distances of the two edges of the stria from the pole. ρ is the relative retardation of the wave front on the two sides of the stria.

We may write $\sigma = \frac{\rho}{x_1 - x_2}$

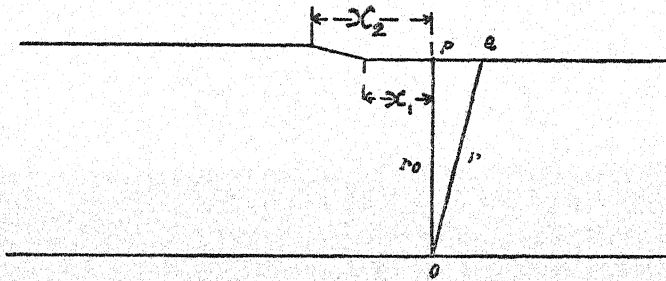


FIG. 9.

We then get as the expression for the amplitude omitting constants

$$\int_{-x_1}^{\infty} \cos 2\pi \left(\frac{t}{\tau} - \frac{r}{\lambda} + \frac{1}{8} \right) dx + \int_{-\infty}^{-x_2} \cos 2\pi \left(\frac{t}{\tau} - \frac{r}{\lambda} - \frac{\rho}{\lambda} + \frac{1}{8} \right) dx \\ + \int_{-x_2}^{-x_1} \cos 2\pi \left(\frac{t}{\tau} - \frac{r}{\lambda} - \frac{\sigma(x_1 - x)}{\lambda} + \frac{1}{8} \right) dx$$

Putting

$$\frac{t}{\tau} - \frac{r_0}{\lambda} + \frac{1}{8} = A$$

$$\frac{\rho}{\lambda} = \theta_1$$

$$\frac{\sigma x_1}{\lambda} - \frac{r_0 \sigma^2}{2\lambda} = \theta_2$$

$$x - r_0 \sigma = z$$

$$-z_1 = -x_1 + r_0 \sigma$$

$$-z_2 = -x_2 + r_0 \sigma$$

the expression for the amplitude becomes

$$\int_{-x_1}^{\infty} \cos 2\pi \left(A - \frac{x^2}{2r_0\lambda} \right) dx + \int_{-\infty}^{-x_2} \cos 2\pi \left(A - \theta_1 - \frac{x^2}{2r_0\lambda} \right) dx$$

$$+ \int_{-z_2}^{-z_1} \cos 2\pi \left(A - \theta_2 - \frac{z^2}{2r_0\lambda} \right) dx$$

which may be transformed into the usual form of Fresnel's integrals by making

$$v_1 = x_1 \sqrt{\frac{2}{r_0\lambda}} \quad v_3 = z_1 \sqrt{\frac{2}{r_0\lambda}}$$

$$v_2 = x_2 \sqrt{\frac{2}{r_0\lambda}} \quad v_4 = z_2 \sqrt{\frac{2}{r_0\lambda}}$$

We get the expression for amplitude to be

$$\cos 2\pi A \int_{-v_1}^{\infty} \cos \frac{1}{2} \pi v^2 dv + \sin 2\pi A \int_{-v_1}^{\infty} \sin \frac{1}{2} \pi v^2 dv$$

$$+ \cos 2\pi (A - \theta_1) \int_{-\infty}^{-v_2} \cos \frac{1}{2} \pi v^2 dv + \sin 2\pi (A - \theta_1) \int_{-\infty}^{-v_2} \sin \frac{1}{2} \pi v^2 dv$$

$$+ \cos 2\pi (A - \theta_2) \int_{-v_4}^{-v_3} \cos \frac{1}{2} \pi v^2 dv + \sin (A - \theta_2) \int_{-v_4}^{-v_3} \sin \frac{1}{2} \pi v^2 dv$$

putting

$$a_1 = \int_{-v_1}^{\infty} \cos \frac{1}{2} \pi v^2 dv, \quad a_2 = \int_{-v_1}^{\infty} \sin \frac{1}{2} \pi v^2 dv$$

$$b_1 = \int_{-\infty}^{-v_2} \cos \frac{1}{2} \pi v^2 dv, \quad b_2 = \int_{-\infty}^{-v_2} \sin \frac{1}{2} \pi v^2 dv$$

$$c_1 = \int_{-v_4}^{-v_3} \cos \frac{1}{2} \pi v^2 dv, \quad c_2 = \int_{-v_4}^{-v_3} \sin \frac{1}{2} \pi v^2 dv$$

We get the following expression for the intensity

$$I^2 = (a_1 + b_1 \cos 2\pi\theta_1 + c_1 \cos 2\pi\theta_2 - b_2 \sin 2\pi\theta_1 - c_2 \sin 2\pi\theta_2)^2$$

$$+ (a_2 + b_1 \sin 2\pi\theta_1 + c_1 \sin 2\pi\theta_2 + b_2 \cos 2\pi\theta_1 + c_2 \cos 2\pi\theta_2)^2$$

In the particular case of a stria whose spectrum of laminary diffraction is illustrated in fig. 4 in the plate,

$$\rho = .00054 \text{ m/m}$$

$$x_1 - x_2 = .025 \text{ m/m}$$

$$r_0 = 1 \text{ meter}$$

the illumination curves for three wave lengths

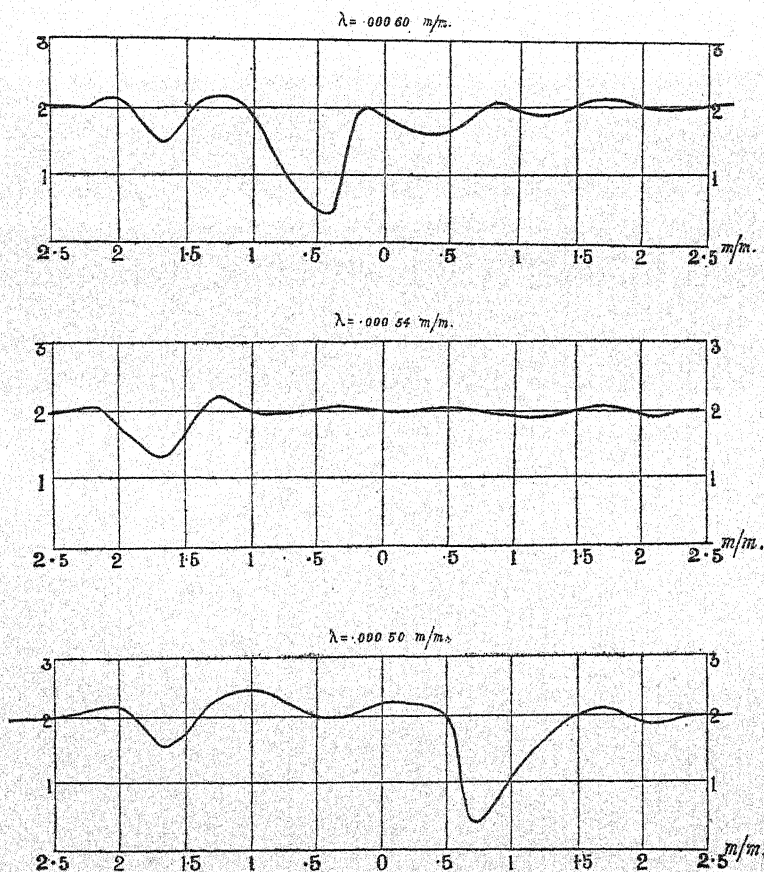
$$\lambda = .00050 \text{ m/m}$$

$$\lambda = .00054 \text{ m/m}$$

$$\lambda = .00060 \text{ m/m}$$

have been calculated and plotted (fig. 10).

Fig. 10.



They clearly show that

- (1) for wave length $\lambda = .00050 \text{ m/m}$ the central dark fringe has somewhat shifted to one side;

- (2) for wave length $\lambda = \cdot 00054$ m/m the central dark fring has vanished ;
- (3) for wave length $\lambda = \cdot 00060$ m/m the central dark fringe has shifted to the opposite side.

This is precisely the effect which appears in fig. 4 in the plate on the two sides of the transmission region in the spectrum.

SECTION VII.—SYNOPSIS.

In a previous paper (published in the Proc. Roy. Soc. for November 1919), the author has described and explained the colours shown by the striae or laminar boundaries in mica when the same is examined by the Foucault test. The present paper describes some observations on the microstructure of these laminar boundaries and of the various diffraction effects produced by them. The following are the principal results obtained.

(a) The striae appear resolved in the microscope into minute echelons or staircase structures, the number of the steps varying from one to ten or fifteen for different striae. The optical retardation due to any particular edge in the echelon is generally quite a small fraction of a wave length. The edges are nevertheless clearly visible in the microscope as very sharp dark lines.

(b) In consequence of the structure of the striae above mentioned, the laminar diffraction pattern observed even at a considerable distance from the mica shows distinct evidence of asymmetry in the distribution of intensity and colour of the fringes.

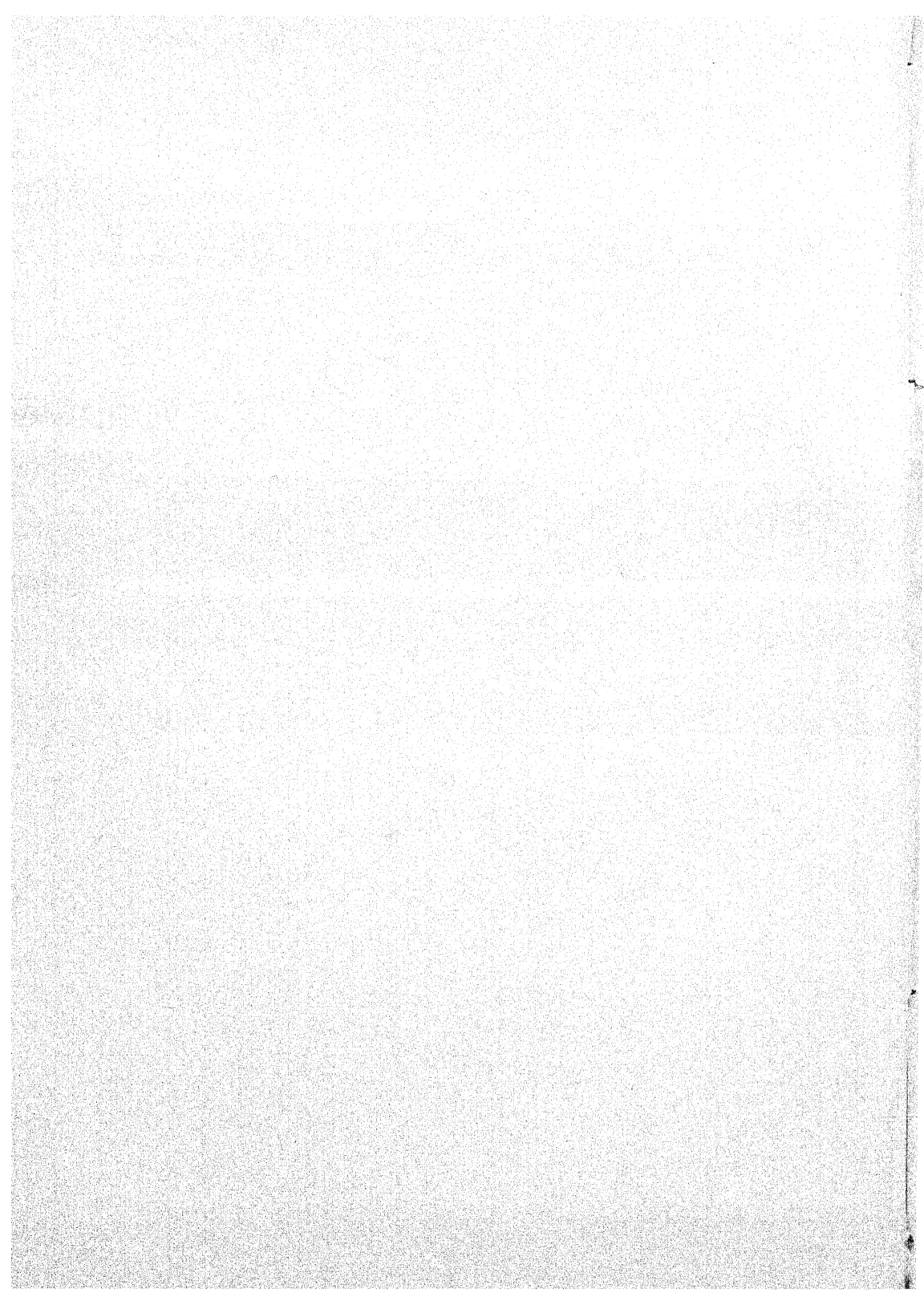
A very delicate method of exhibiting this asymmetry is furnished by spectroscopic analysis of the laminar diffraction pattern, the dark bands due to interference running obliquely through the spectrum and being much more clearly marked on one side of the pattern than on the other.

(c) In the immediate neighbourhood of the striae the diffraction-phenomena as observed through a microscope are more complicated, the asymmetry being very marked, and a very large number of fringes may be observed.

(d) The striae scatter light through large angles, the light thus diffracted showing a marked asymmetry in its intensity on the two sides of the direction of the incident light and also exhibiting both colour and polarisation.

The investigation described in this paper was carried out in the Palit Laboratory of Physics, and the author's best thanks are due to Prof. C. V. Raman for his suggestions and unfailing interest in the work during its progress.





V. On the Forced Oscillations of Strings under Damping proportional to the Square of the Velocity.

By Rajendra Nath Ghosh, M.Sc., Research Scholar of the Association.

(Plate V.)

CONTENTS.

- SECTION I.—Introduction.
SECTION II.—Law of Damping of Free Vibrations of Strings.
SECTION III.—Experimental Study of Forced Vibrations of Strings.
SECTION IV.—Theory of Forced Oscillations under Transverse Excitation.
SECTION V.—Theory of Forced Oscillations under Longitudinal Excitation.
SECTION VI.—Synopsis.

SECTION I.—INTRODUCTION.

In a recent paper published in the *Physical Review** J. Parker Van Zandt has given a discussion and comparison with experiment of the theory of *free* oscillations of bodies subject to resisting forces proportional to the *square* of the velocity. He has also given a very full bibliography of the literature on oscillations subject to this and other special laws of damping. In the course of some quantitative work on the forced oscillations of strings under different types of excitation recently undertaken by the present author, results have been obtained which show that the frictional forces acting on a stretched string vibrating in air are not propor-

* *Physical Review*, Nov. 1917, page 415.

tional to the velocity of each point on the string as is generally assumed,* but increase much more rapidly than in proportion to the velocity. It accordingly appeared to be of interest to investigate the theory of forced oscillations under damping proportional to the square of the velocity, and to compare the results with those found in experiments on stretched strings vibrating under different types of excitation. This has been done, and the present paper describes the outcome of the investigation. The experimental work has been carried out with the aid of the apparatus for the study of vibrations of strings developed by Prof. C. V. Raman and described in a recent publication.† This apparatus is specially suitable for the present investigation and has indeed made it possible.

SECTION II.—LAW OF DAMPING OF FREE VIBRATIONS OF A STRETCHED STRING.

Observations have been made by the author of the rate of decay of the free oscillations of a stretched cord (of twisted cotton thread) vibrating in air. This was investigated by a photographic method. The middle point of a stretched string was pulled aside and then released. The damped oscillation of the point was photographed on a moving plate, and measurements of the amplitudes were made by means of a cross-slide micrometer. The following table shows a typical set of results. In column I we have the successive amplitudes, in column II their successive differences D , in column III the mean M of two successive amplitudes.

* Lord Rayleigh, "Theory of Sound," Vol. I, Arts. 131-134.

† C. V. Raman, "An Experimental Method for the Production of Vibrations," *Physical Review*, Nov. 1919.

TABLE I.
Length of String 100 cms.

No.	Successive amplitudes.	D	M	D/M ²	D/M
1	cms. 2·0344	·3559	1·8564	·13	·19
	1·6785				
2	1·4102	·2683	1·5443	·13	·17
3	1·2115	·1982	1·3108	·11	·15
4	1·0363	·1752	1·1239	·13	·15
5	0·9069	·1294	0·9716	·14	·13
6	0·8147	·0922	0·8608	·12	·11
7	0·7050	·1097	0·7568	·13	·14
8	0·6424	·0626	0·6737	·14	·09

In the fourth column it will be observed that D/M^2 is almost constant whereas D/M shows a variation much greater than the experimental errors. Now we know that when the damping is proportional to the first power of the velocity, the ratio of two successive amplitudes is constant, from which it can be easily shown that the difference of two successive amplitudes divided by their mean must also be a constant. But in the present case it is seen that the above ratio namely D/M is not constant, while D/M^2 is constant. Hence the law of frictional resistance is not that of the first power of the velocity. We shall presently show that the result D/M^2 is a constant expresses the fact that the frictional forces are proportional to the square of the velocity.

The equation of motion of a particle on the stretched string vibrating freely and resisted by forces proportional to the square of the velocity is given by

$$\ddot{U} = -n^2 U \pm k\dot{U}^2 \quad \dots \dots \dots (1)$$

where U is the displacement and the constants have their usual meanings. For a first approximation, neglect the term $\pm k\dot{U}^2$, and we have

$$U = A \cos nt \quad \dots \dots \dots (2)$$

By the method of successive approximation we get the value of U when resisting forces are taken into consideration.

$$U = A \cos nt + \frac{1}{2} A^2 k + \frac{1}{8} A^2 k \cos 2nt \quad \dots \quad (3)$$

Let $U = A_1$ and $\frac{dU}{dt} = 0$ when $t = 0$

Therefore we get $A = A_1 - \frac{2}{3} k A_1^2$ and U is then given by

$$U = (A_1 - \frac{2}{3} k A_1^2) \cos nt + \frac{1}{2} A_1^2 k + \frac{1}{8} A_1^2 k \cos 2nt \quad \dots \quad (4)$$

Equation (4) is only true for half an oscillation. At $t = \pi/n$ the displacement U is given by

$$U = -(A_1 - \frac{2}{3} A_1^2 k)$$

Hence the amplitude of swing is diminished by $\frac{2}{3} A_1^2 k = D$ (approx.). Hence we arrive at once at the result that

$$D/M^2 = \frac{2}{3} k \quad (5)$$

From the experimental results we have seen that D/M^2 is almost a constant. Hence we infer that the frictional forces for the large amplitudes of vibration employed are proportional to the square of the velocity.

From other results obtained by the author it would seem that for very small amplitudes of vibration on the other hand, the frictional forces do not increase quite so rapidly as in proportion to the square of the velocity, and are more nearly proportional to the first power.

SECTION III.—EXPERIMENTAL STUDY OF FORCED VIBRATIONS OF STRINGS.

Description of apparatus.

For an experimental study of the forced oscillations of stretched strings, an electric motor-vibrator was used. A detailed account of this apparatus as originally devised by Professor J. A. Fleming, and subsequently modified and improved by Professor C. V. Raman, has been published in the *Physical Review*.* In Plate V, Fig. 1 shows the general features of the apparatus. A circular wheel is fixed to the axle of a small motor. To this wheel a brass disk carrying a slot and movable pin is fixed. The pin

* November, 1919.

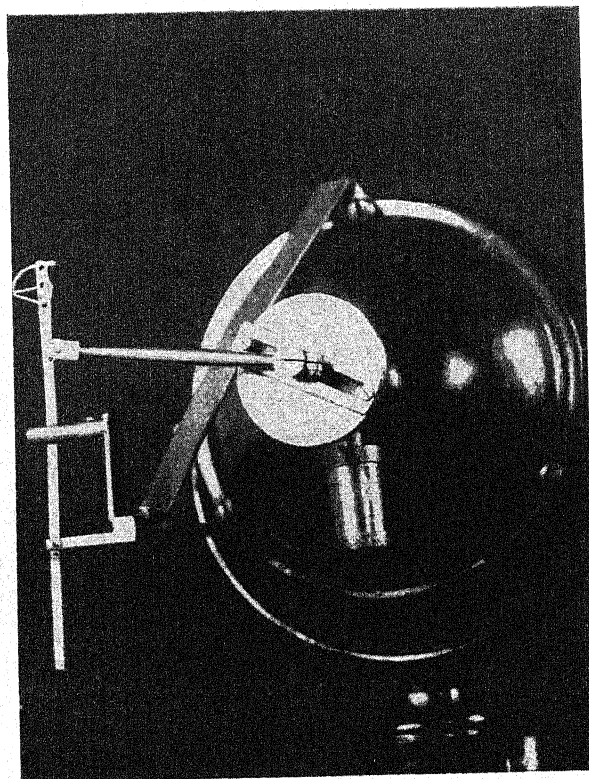


Fig. 1

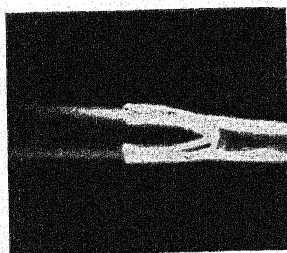


Fig. 2

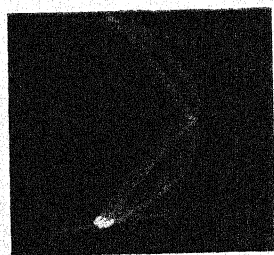
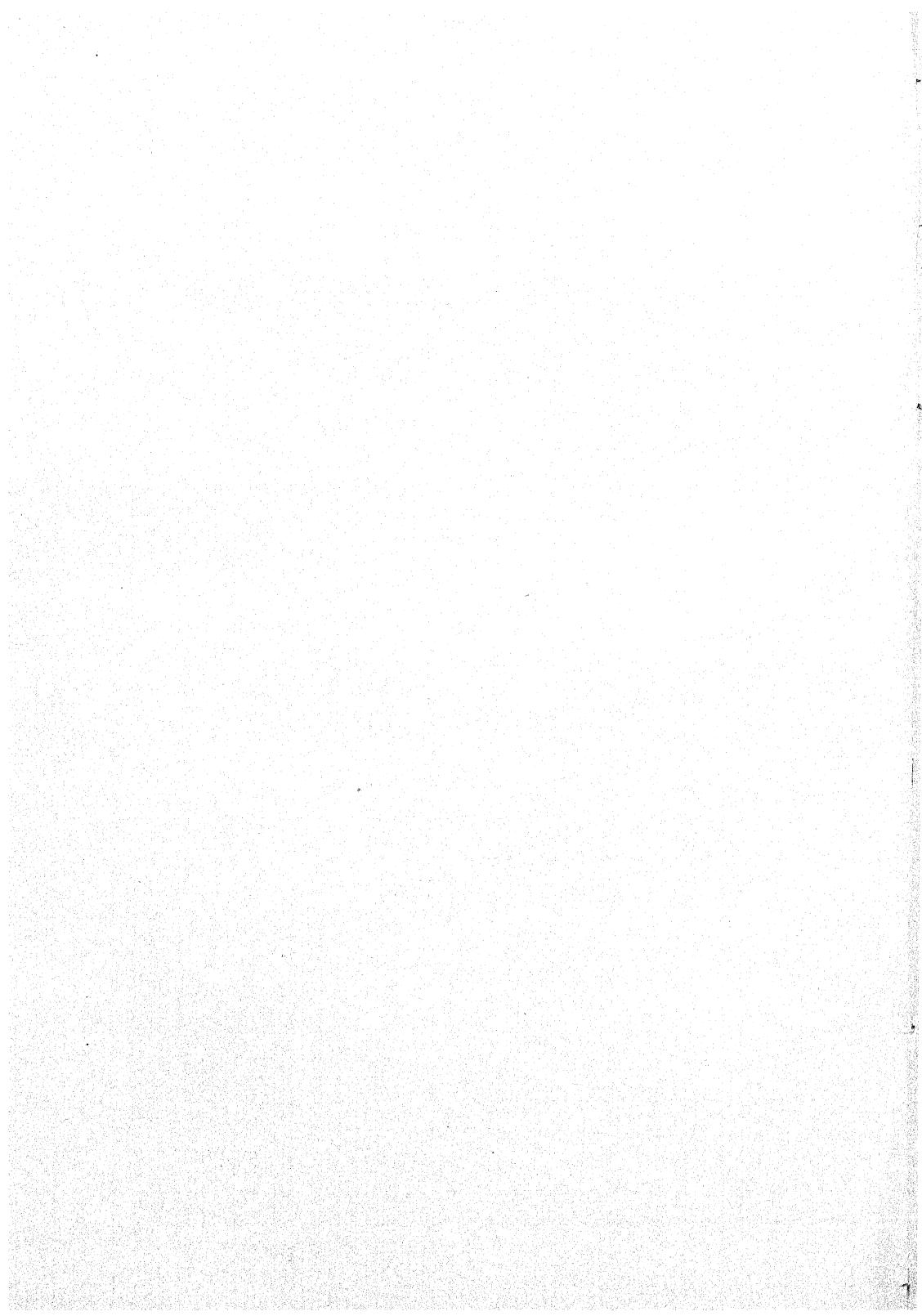


Fig. 3



actuates a crank shaft and oscillating lever moving between guides. The oscillating lever performs what is approximately a simple harmonic motion, and excites the vibration of the string in the manner of Melde's experiment. The frequency of this excitation depends upon the rate of revolution of the motor; and the amplitude of motion of the lever-arm can be governed to a nicety by altering the position of the pin on the motor. When the pin is exactly at the centre, the amplitude of motion of the lever is zero, and as the pin is moved away from the centre, the amplitude increases. This adjustability of the amplitude enhances the utility of the apparatus. The frequency of the vibrator, i.e. of the lever arm, was kept constant by a sliding rheostat and the constancy was observed through a stroboscopic tuning fork. In all the experiments, the frequency was generally 60 per second. The tension was adjusted by putting weights on a pan and clamping the end of the string near it, so that the pan could not oscillate. The motor-vibrator permits the use of long strings, considerable tensions, and gives a perfectly constant amplitude, advantages which are not possible when a tuning fork is used as the exciter.

The amplitude of vibration of the string depends on (1) its tension and (2) the magnitude of the obligatory motion imposed on it at the end. Keeping (2) constant, (1) may be varied and the resonance curve of the string may be traced. This may be done for a series of values of (2).

Transverse Excitation.

The case first studied in the manner described above was the transverse type of Melde's experiment in which the maintained oscillation has the same period as the obligatory motion impressed at one end. This has been discussed mathematically by the late Lord Rayleigh * on the assumption that the frictional resistances are proportional to the first power of the velocity. Corresponding to an obligatory motion $Y = \gamma \cos pt$ at a nodal point, the amplitude of vibration of the string at any point is approximately given by

$$A_x = \gamma \left\{ \frac{\sin^2 \frac{2\pi}{\lambda} x + \frac{kx^2}{4a^2} \cos^2 \frac{2\pi}{\lambda} x}{\frac{k^2 b^2}{4a^2}} \right\}^{\frac{1}{2}},$$

* Theory of Sound, Vol. I, Art. 134.

so that we should find the amplitude of vibration to be proportional to the obligatory motion γ . Actually, however, on determining the amplitude of vibration of the string for the different values of

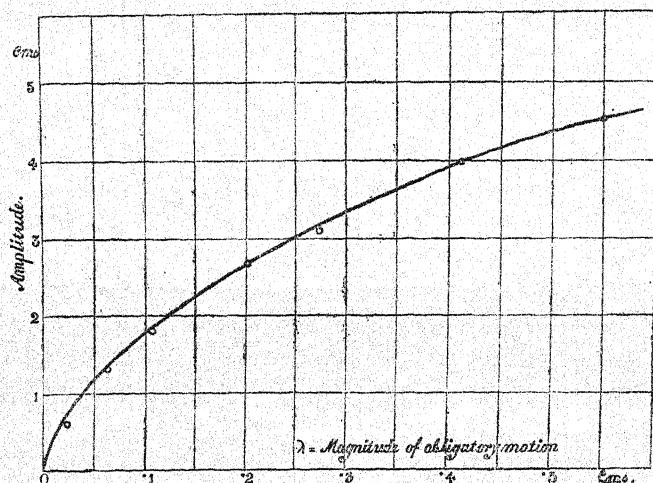


FIG. 4.—Relation between Amplitude and Obligatory Motion.

γ , the amplitude increases much less rapidly than in proportion to γ . The experimental results are shown in fig. 4 in which the

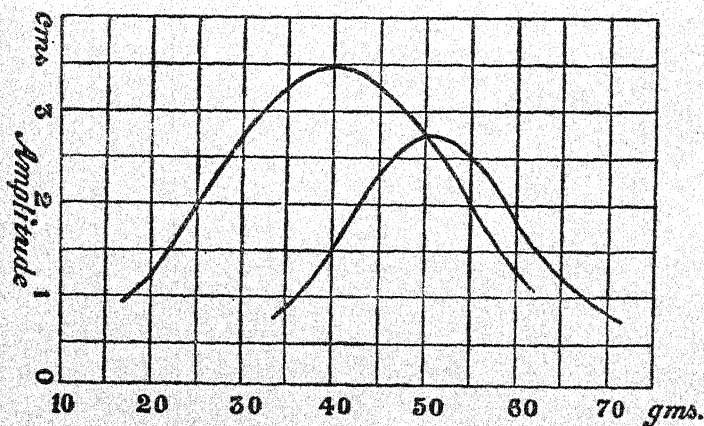


FIG. 5.—Resonance Curves for two different values of γ .

ordinates represent the maximum amplitudes of vibration obtainable with the respective extents of the obligatory motion shown as abscissae. The form of the graph is approximately a parabola.

This result indicates that the frictional resistance is not proportional to the first power of the velocity, but, as I shall show later on, is more nearly in proportion to the square of the velocity.

Another important fact noticed was that the tension for which the maximum amplitude is obtained decreases with increasing values of γ . This point was studied by adjusting the tension of the string, clamping its free end, and then starting the motor vibrator. The experimental results are shown in fig. (5) where the amplitudes of vibration of the string are plotted against the tension for two different values of γ . The shift of the maximum amplitude and of the whole curve when the amplitude of vibration is large, is evident from a comparison of the two curves. The theoretical interpretation of this result will be discussed later in the course of the paper.

Longitudinal Excitation.

The next case studied was the longitudinal type of Melde's experiment in which the obligatory motion imposed at one end varies the tension of the string. In this case, as has been shown by C. V. Raman,* maintenance is possible when the frequencies of the obligatory motion and of the oscillation maintained stand in any of the ratios $2 : n$ where n is an integer. In the present paper attention will be confined to the first case in which the frequency ratio is $2 : 1$. The theory of this case was first given by the late Lord Rayleigh, and later modified and extended by Prof. Raman in the papers cited so as to give results more in accordance with experiment, the assumption being made that the frictional force is proportional to the first power of the velocity. The experimental observations of Prof. Raman were made using a tuning fork as exciter, and as the reaction of the string on the fork materially influences the vibration of the fork and alters its amplitude, a difficulty arises in interpreting the results observed. With the electric motor-vibrator, on the other hand, the amplitude of the obligatory motion is invariable, and its frequency can be kept constant by the use of a rheostat so that the reaction of the string can exercise no effect. A stricter comparison of the

* Bulletin No. 6 of the Association, also *Phil. Mag.*, October 1912, and *Physical Review*, December 1912, July 1914 and November 1919; see also Jones and Phelps, *Physical Review*, November 1917.

experimental results with theory is therefore possible. The experimental observations may be considered under two heads, (a) dependence of amplitude of vibration upon the tension and the magnitude of its periodic variation, and (b) dependence of the phase of vibration on the same variables.

(a): In order to study the relation between the amplitude of vibration of the string and the tension, the latter was made very high at the beginning, and then it was continuously diminished. With the change in tension, the speed of the motor and of the variation of tension produced by it tend to change to a slight extent, but they were kept constant as explained above

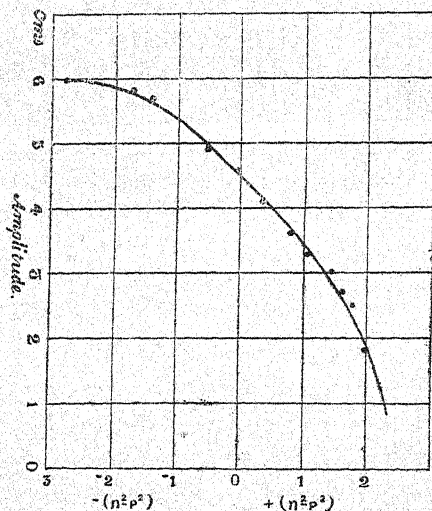


FIG. 6.—Resonance Curve under Variable Spring of Double Frequency.

with the aid of a sliding rheostat and a stroboscopic fork. It was found that at a high tension (much greater than the value for which the natural frequency of vibration of the string is half that of the vibrator) the amplitude of vibration is small, but has always a definite value for a definite tension. As the tension is diminished, the amplitude of vibration increases, and continues to increase till a stage is reached when the tension is much lower than the theoretical value for which the natural frequency is half that of the vibrator; the motion of the string then suddenly collapses giving place to other types of oscillation. At this instant the amplitude is a maximum. When the tension is such that

the natural frequency is half that of the vibrator, the amplitude is much less than the maximum value. The amplitude is not the same when the tension is greater or less than the theoretical value for half frequency by a certain amount. The whole sequence of events is shown in fig. 6. The graph shown there has been drawn from the experimental data. The continuous increase of the amplitude with the diminution of tension, and the consequent asymmetric character of the curve are both clearly shown. The largest value of the amplitude obtained at the collapse point is shown by one extremity of the curve.

As the amplitude of the vibrator and consequently also the magnitude of the periodic variation of tension are increased, the maximum possible amplitude of vibration of the string also increases. The experimental curve simply shifts up through a certain distance. If on the other hand we diminish the magnitude of the imposed periodic variation of tension, the amplitude maintained also diminishes, and when the imposed variation falls below a certain very small limit no maintenance is obtained.

(b): The phase relation between the exciting force and the vibration of the string was easily studied by attaching a bright bead to the string near the vibrator. Each point of the string has two motions at right angles to each other, the longitudinal motion being due to that of the vibrator, and the other the transverse motion of the string itself. Hence the Lissajous figure traced by the bead gives the phase relation required. Using arc-light to illuminate the bead, the Lissajous figures were photographed at different amplitudes of vibration of the string. Fig. 2 in Plate V shows the figure when the amplitude was small corresponding to a high tension. The parabolic figure seen (convex towards the vibrator) at once indicates the frequency and phase relations between the longitudinal and transverse components of the motion, in other words between the maintaining force and the maintained vibration. Fig. 3 in the Plate shows the Lissajous figure when the amplitude of vibration of the string was large corresponding to a low tension. The change in phase which has occurred in consequence of decrease of tension is evident. The rapidity with which the change occurs depends upon the magnitude of the imposed variation of tension. When this is large, we find that the phase relation alters slowly till the form of the

Lissajous figure which is originally a parabola convex to the vibrator reaches the limiting form of an 8 curve. At this stage the vibration collapses. On the other hand when the magnitude of the periodic variation of tension is small, the phase changes rapidly with decrease of tension, the initial and final forms of the Lissajous figures are the parabola and the 8 curve respectively in the same way as in the case previously mentioned.

SECTION IV.—THEORY OF FORCED OSCILLATION UNDER TRANSVERSE EXCITATION.

In this section we shall deal with (1) the forced vibration of a simple system of one degree of freedom, (2) the forced vibration of a stretched string at a point of which an obligatory motion is imposed, the damping in either case being assumed to be proportional to the square of the velocity.

Case (1). The equation of motion of a system of one degree of freedom, subject to frictional forces proportional to the square of the velocity, and acted on by a periodic force $a \sin pt$, is given by

$$\ddot{U} \pm k \dot{U}^2 + n^2 U = a \sin pt \quad (6)$$

the alternative signs being necessary as otherwise the frictional forces do not change sign with the velocity.

The solution of this equation will be of the form

$$U = A_1 \sin(pt + \epsilon_1) + A_3 \sin(3pt + \epsilon_3) + A_5 \sin(5pt + \epsilon_5) + \dots \quad (7)$$

Substituting (7) in (6) we get the resistance term as

$$\pm k \dot{U}^2 = \pm k p^2 \{ A_1^2 \cos^2(pt + \epsilon_1) + 6 A_1 A_3 \cos(pt + \epsilon_1) \cos(3pt + \epsilon_3) + 9 A_3^2 \cos^2(3pt + \epsilon_3) + \dots \} \quad (8)$$

Retaining terms proportional to $\pm A_1^2$ and applying the Fourier analysis to get the periodic components of the frictional force, we get

$$\begin{aligned} \pm k \dot{U}^2 = \frac{2}{\pi} k p^2 A_1^2 \left\{ \frac{4}{3} \cos(pt + \epsilon_1) - \frac{4}{15} \cos(3pt + 3\epsilon_1) \right. \\ \left. - \frac{4}{105} \cos(5pt + 5\epsilon_1) \right\} \quad (9) \end{aligned}$$

The series in (9) is a rapidly converging one. Substituting (9) in (6), expanding and equating the coefficients of $\sin pt$ and $\cos pt$ to zero, we get the following relations:—

$$A_1^2 = \frac{(\Delta^4 + 4\alpha^2 k_1^2 p^4)^{\frac{1}{2}} - \Delta^2}{2k_1^2 p^4} \quad (10)$$

$$\tan E_1 = - \left\{ \frac{(\Delta^4 + 4\alpha^2 k_1^2 p^4)^{\frac{1}{2}} - \Delta^2}{2\Delta^2} \right\} \quad (11)$$

where $\Delta = n^2 - p^2$ and $k_1 = \frac{8k}{3\pi}$.

The energy of the forced oscillation is a maximum when $\Delta = 0$, i.e. when $n = p$. The amplitude is then given by

$$A_1 = \frac{1}{p} \left(\frac{\alpha}{k_1} \right)^{\frac{1}{2}} \quad (12)$$

These expressions show that the amplitude of the maintained oscillation at the peak of the resonance curve is proportional to the square root of the magnitude of the impressed force instead of the first power as in the ordinary case. Another special feature of the forced oscillation for this law of resistance is that the phase of the maintained motion, except at the peak of the resonance curve, varies with the magnitude of the impressed force, as can be seen from equation (11).

Defining *sharpness of resonance* as the square root of the quotient of the curvature at the peak of the resonance curve*, divided by the energy at the same position, we get

$$\text{Sharpness of resonance} = \frac{n}{(k_1 \alpha)^{\frac{1}{2}}}$$

So that we find that the sharpness of resonance is directly proportional to the frequency of oscillation of the system, and inversely proportional to the square root of the product of the impressed force and the damping coefficient, contrary to the case of ordinary resonance in which the sharpness of resonance defined in the same way is given by the equation †

$$\text{Sharpness of resonance} = \frac{n}{k_1}$$

We have seen that the resonance is maximum when $n = p$,

* The energy should be graphed as a function of the mistuning $\left(\frac{x}{p} - \frac{p}{n} \right)$.

† E. H. Barton, *Phil. Mag.*, July 1913.

but the amplitude of maintained motion is maximum when the frequency of the impressed force is equal to

$$p_1 = (n^2 - k_1 a)^{\frac{1}{2}} \text{ (approx.)} \quad (13)$$

and is therefore less than the natural undamped frequency of the system. The difference of the squares of the frequencies natural to the system and that for which the amplitude is a maximum, is approximately proportional to the product of the damping coefficient and the impressed force, whereas in the ordinary case, the difference of the squares of the above frequencies is proportional to the square of the damping coefficient.

The maximum amplitude is given by

$$A_1 = \left\{ \frac{a}{k_1 (n^2 - k_1 a)} \right\}^{\frac{1}{2}} \text{ approximately} \quad (14)$$

which shows that the maximum amplitude is proportional to the square root of the impressed force.

When the amplitude of vibration is large and the restoring force cannot be taken as strictly proportional to the displacement, we may modify the equation of motion, and write

$$\ddot{U} \pm k\dot{U}^2 + (n^2 + \beta U^2) U = a \sin pt \quad (15)$$

The displacement will be given by a series of the form

$$U = A_1' \sin (pt + \epsilon_1') + A_3' \sin (3pt + \epsilon_3') + A_5' \sin (5pt + \epsilon_5')$$

Applying the same method of analysis and proceeding as before we get equations similar to (10) and (11) etc., the only change necessary being made by writing Δ_1 for Δ in all the equations, where

$$\Delta_1 = n^2 - p^2 + \frac{3}{4} \beta A_1'^2$$

The effect of a finite amplitude is therefore equivalent to an increase in the natural frequency of oscillation of the system. Hence the maxima of all the resonance curves will not be found when $n=p$, but for large amplitudes, they will shift towards a lower natural frequency of vibration.

Case (2). Now we shall pass on to the case of forced oscillation of a stretched string whose every elementary portion is resisted by forces proportional to the square of the velocity. The equation of motion of the string is given by

$$\frac{d^2 y}{dt^2} \pm k \left(\frac{dy}{dt} \right)^2 = a^2 \frac{d^2 y}{dx^2} \quad (16)$$

the alternative signs being necessary as explained before. The general solution of (16) is

$$y = u_1 \sin pt + v_1 \cos pt + u_3 \sin 3pt + v_3 \cos 3pt + \text{etc.} \quad (17)$$

where the u 's and v 's are functions of x only.

Substituting (17) in (16) we get the frictional force of the form

$$\pm k p^3 \{ u_1^2 \cos^2 pt + v_1^2 \sin^2 pt + \dots \}, \text{ (approx.)}$$

Retaining terms proportional to u_1^2 and v_1^2 and analysing in Fourier series, we get

$$\begin{aligned} & \pm k p^3 \{ u_1^2 \cos^2 pt + v_1^2 \sin^2 pt \} \\ & = -\alpha p^2 \{ u^2 \cos pt + v^2 \sin pt + \text{terms of higher frequency} \} \quad (18) \end{aligned}$$

$$\text{where } \alpha = \frac{8k}{3\pi}.$$

Substituting (18) in (16) and equating the coefficients of $\sin pt$ and $\cos pt$ to zero, we get

$$\left. \begin{aligned} \alpha^2 \frac{d^2 v}{dx^2} + p^2 v + \alpha p^2 u^2 &= 0 \\ \alpha^2 \frac{d^2 u}{dx^2} + p^2 u + \alpha p^2 v^2 &= 0 \end{aligned} \right\} \quad (19)$$

The approximate solutions of (19) are

$$\begin{aligned} u_1 &= A_1 \sin px/a - B_1^2 a/2 + B_1^2 a/6 \cos 2 px/a + \dots \\ v_1 &= B_1 \sin px/a - A_1^2 a/2 + A_1^2 a/6 \cos 2 px/a + \dots \end{aligned}$$

from $x=0$ to $x=b$.

$$\therefore y = R_x \sin (pt + \phi x)$$

$$\text{where } \tan \phi_x = \frac{B_1 \sin px/a - A_1^2 a/2}{A_1 \sin px/a - B_1^2 a/2} = \frac{f_1}{f_2} \text{ approx.}$$

and

$$R_x^2 = f_1^2 + f_2^2.$$

From the conditions that at $x=b$, $y=\gamma \sin pt$ the values of A_1 and B_1 are determined. Let us now take the important case in which the period of the forced vibration equals the natural period of the string; then $\sin pb/a=0$ and all the expressions are much simplified. The expression for the displacement then comes out to be approximately

$$Y = \sqrt{\frac{2\gamma}{a}} \sin px/a \cdot \sin (pt + \phi_x)$$

where

$$\tan \phi_x = - \left(\frac{2}{\gamma a} \right)^{\frac{1}{2}} \sin px/a \quad (20)$$

From (20) we see that the amplitude of vibration of the string is proportional to the square root of the obligatory displacement unlike the case when friction is proportional to the velocity, where the amplitude of vibration is directly proportional to the obligatory displacement. The motion of all points on the string except near the nodes is in approximately the same phase. The phase of the motion at the nodes is the same as that of the obligatory motion, while that of the rest of the string lags behind by quarter of an oscillation. The phase changes rapidly near the nodes, and two points of the string on opposite sides of the node at a sufficient distance from it generally move in opposite directions.

The experimental results represented by fig. (4) agree with the theory according to which the maximum amplitude of vibration at a point for different values of γ increases much less rapidly than in proportion to it.

The effect of the finite amplitude of vibration can be easily calculated. The total length of the vibrating string at any instant is given by the formula

$$\begin{aligned} l &= \int_0^l dx \left\{ 1 + \left(\frac{dy}{dx} \right)^2 \right\}^{\frac{1}{2}} \\ &= \int_0^l dx \left\{ 1 + \frac{1}{2} \left(\frac{dy}{dx} \right)^2 \right\} \\ &= l_0 + \frac{1}{2} \int_0^l \left(\frac{dy}{dx} \right)^2 dx \end{aligned}$$

Whatever may be the nature of the function $\left(\frac{dy}{dx} \right)$, the effect of the finite amplitude is to increase the length. If, however, there be no longitudinal motion, it produces an increase in tension which would amount to a large value when the modulus of elasticity is large. This increase in tension will therefore bring the points of maximum resonance towards lower tension and shift the whole resonance curve. This is the result actually obtained

as will be found from a reference to the experimental curve given in fig. (5).

SECTION V.—THEORY OF FORCED OSCILLATIONS UNDER LONGITUDINAL EXCITATION.

We shall now pass on to the case of forced vibration maintained by forces of double frequency. The theory of oscillations under a periodic force varying the 'spring' was first given by the late Lord Rayleigh* on the assumption that the resisting force is proportional to the velocity. The theory has since been extended by Prof. C. V. Raman† who has shown how the equations may be modified and solved so as to give a determinate amplitude for the motion.

CASE (I).—Damping Proportional to the Velocity.

This has been dealt with by Prof. Raman in the papers cited. The equation of motion of an oscillation of one degree of freedom under variable spring and subject to damping proportional to the velocity is

$$\ddot{U} + k \dot{U} + (n^2 - 2a \sin 2pt + \beta U^2) U = 0$$

The solution for the case in which the force has double the frequency of the maintained motion is

$$U = A_1 \sin(pt + \epsilon_1) + A_3 \sin(3pt + \epsilon_3) + A_5 \sin(5pt + \epsilon_5) + \text{etc.}$$

where A_1 and ϵ_1 are given as a first approximation, by

$$(n^2 - p^2 + \frac{3}{4}\beta A_1^2)^2 = a^2 - k^2 p^2 \quad (21)$$

and

$$\tan \epsilon_1 = \left(\frac{\alpha - kp}{\alpha + kp} \right)^{\frac{1}{2}} \quad (22)$$

CASE (II).—Damping Proportional to the Square of the Velocity.

The equation of motion for this case is

$$\ddot{U} \pm k \dot{U}^2 + (n^2 - 2a \sin 2pt + \beta U^2) U = 0.$$

We assume the solution to be given by

$$U = A_1 \sin(pt + \epsilon_1) + A_3 \sin(3pt + \epsilon_3) + A_5 \sin(5pt + \epsilon_5) + \text{etc.}$$

* Scientific Papers, Vol. II, page 188.

† Bulletin No. 6 of the Indian Association for the Cultivation of Science; also *Physical Review*, Dec. 1912.

Proceeding in the same way as before, we obtain the following relations as a first approximation

$$(n^2 - p^2 + \frac{3}{4}\beta A_1^2)^2 = a^2 - \frac{64}{9\pi^2} k^2 A_1^2 p^4 \quad (23)$$

$$\tan \epsilon_1 = \left(\frac{a - \frac{8}{3\pi} k A_1 p^2}{a + \frac{8}{3\pi} k A_1 p^2} \right)^{\frac{1}{2}} \quad (24)$$

It is important to notice the points of agreement and difference between the results in the two cases.

The amplitude of vibration is not symmetrical with respect to $n^2 - p^2$ in both the cases. In case (I), that is with damping proportional to the velocity, we find that the amplitude of vibration increases continuously as $(n^2 - p^2)$ diminishes. This continues with $(n^2 - p^2)$ negative, and equation (21) does not put any limit to the continuous decrease of $(n^2 - p^2)$ with consequent increase of the amplitude of vibration. It indicates that the tension can be diminished indefinitely, and the amplitude will then attain a correspondingly large value. But equation (23) of case (II) puts a limitation to the indefinite increase of the amplitude corresponding to an infinitely small tension. It shows that the amplitude of vibration is small when the tension is high, and increases with diminution of tension. The maximum value of the amplitude is not reached when $(n^2 - p^2) = 0$, i.e. when the natural frequency of oscillation is half that of the maintaining force. When $(n^2 - p^2)$ becomes negative (the natural frequency of oscillation of the string is less than half that of the maintaining force) the amplitude increases still further. But the rate of increase of the amplitude in case (II) is not so rapid as in case (I). This is due to the fact that the right side of (23) goes on diminishing with the increase of the amplitude whereas the right side of (21) remains constant. This goes on till the right side of (23) becomes zero. At this stage the motion collapses, since the supply of energy becomes less than that lost by dissipation. The maximum value of the amplitude is, therefore, obtained by putting the right side of (23) equal to zero; it is then given by

$$A_m = \frac{3\pi a}{8kp^2} \quad (25)$$

The same result may be obtained in another way. Solving equation (23) we have

$$A_1^2 = \frac{(l^2 + 4\beta^2 m)^{\frac{1}{2}} - l}{2\beta^2}$$

where

$$l = 2\Delta\beta + k_2 \text{ and } m = a^2 - \Delta^2$$

and

$$\Delta = n^2 - p^2 \text{ and } k_2^2 = \frac{64}{9\pi^2} k^2 p^4$$

The value of Δ when A_1^2 is maximum is given by

$$\frac{\partial A_1^2}{\partial \Delta} = 0 \text{ which gives } \Delta = -\frac{a^2\beta}{k_2}$$

The maximum value of A_1 is thus as before equal to

$$A_1 = \frac{3\pi a}{8kp^2}$$

Thus we find in case (II) there is a limit to the increase of the amplitude with diminution of $n^2 - p^2$ whereas in case (I), equation (21) does not put any limit to the increase of the amplitude.

The foregoing indications of theory agree closely with the observed facts. The continuous increase of the amplitude with diminution of tension has been found to be the same as required by the theory. The theoretical value of the amplitude when $n=p$ is the same as found from experiment.

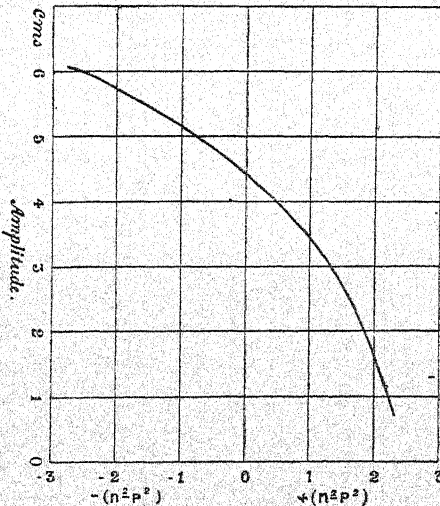


FIG. 7.—Resonance-Curve for Variable Spring of Double Frequency with Damping proportional to the Square of the Velocity.

The theoretical graph fig. (7) plotted from equation (23) shows the relation between the amplitude of vibration of the string and $n^2 - p^2$. The values of α and $k\phi$ required for drawing the theoretical graph were determined directly from the known elasticity and damping of the string, and from the amplitude of the obligatory motion of the end of the string and the frequency of the motor vibrator. The agreement of the curves in figs. (6) and (7) is close. The amplitude is not symmetrical with respect to $n^2 - p^2$ as can be seen from the figures. The values of maximum amplitudes agree in the two curves. When $n=p$, the amplitude is given by the point where the curve cuts the axis representing the amplitude. Both the curves agree here also. The rate of increase of the amplitude is not the same everywhere in fig. (7): it is less near the collapse point than at small amplitudes of motion in accordance with the observed fact indicated by fig. (6).

Equation (21) of case (I) expresses the fact that in order to maintain the vibratory motion of the string, α must be greater than $k\phi$ whatever the values of n and ϕ , whereas equation (23) does not fix any minimum value for α . An attempt was made to maintain the motion of a long string with a very small magnitude of α , but it appeared that no maintenance was possible. This indicates that the frictional resistance for very small amplitudes is proportional to the first power of the velocity, and equations (21) and (22) then hold good; as soon as the amplitude of vibration of the string becomes sensible, the observed facts are better in agreement with the theory worked out on the basis that the damping forces are proportional to the square of the velocity.

The changes in the phase relation between the maintaining force and the maintained motion of the string may now be explained. In case (I), equation (22) shows that when α , k and ϕ are constant, i.e. when the amplitude of the motor-vibrator, its frequency, and the frictional coefficient are constant, the phase angle ϵ should remain constant, whereas equations (23) and (24) of case (II) show that the phase angle will change even when the above quantities are constant if the permanent tension of the string and consequently its amplitude of vibration are altered. Equation (24) shows that as the tension is gradually decreased, at the initial stage where the amplitude of vibration is small, the phase angle varies very slowly, but when the amplitude is considerable, then it begins to decrease rapidly, and it continues

to diminish, and attains the limiting value when $A_1 = \frac{3\pi a}{8k\rho^2}$, i.e. when the motion of the string collapses. This is in agreement with observation.

SECTION.—SYNOPSIS.

1. For large amplitudes of vibration it has been found that the frictional resistance to the vibration of a string in air is approximately proportional to the square of the velocity.

2. The forced vibration of a simple vibrator resisted by forces proportional to the square of the velocity has been dealt with, and it has been found (1) the maximum resonance is proportional to the square root of the impressed force, (2) the sharpness of resonance is directly proportional to the frequency natural to the system, and inversely proportional to the square root of the product of the frictional coefficient and the impressed force.

3. In the vibrations of a string obtaining by imposing a transverse obligatory motion at one point it is found that (1) the maximum amplitude of vibration increases much less rapidly than in proportion to the obligatory displacement. This has been theoretically accounted for by the square law of damping; (2) the effect of finite amplitude of vibration in shifting the resonance curve towards lower tensions has been shown by graphs and explained.

4. In the longitudinal type of Melde's experiment when the frequency of oscillation maintained is half that of the imposed variation of tension, it is found experimentally that (a) there is a definite limit corresponding to a particular magnitude of variation of tension imposed on the string, beyond which the amplitude of vibration cannot be increased by decreasing the constant part of the tension, the maintenance collapsing when this stage is reached and passed; and (b), that the phase relation between the maintaining force and the maintained oscillation varies in different parts of the range of maintenance even when the imposed variation of tension is kept constant in frequency and magnitude. These facts cannot be fully explained if the frictional force resisting the motion of the string is assumed to be proportional to the first power of the velocity, but agree with the theory developed on the assumption that the friction is proportional to the square of the velocity.

5. The experimental work described in the paper was carried out by the aid of the motor-vibrator designed by Prof. C. V. Raman, which was found specially suitable for the present investigation and indeed made it possible.

6. The author hopes on a future occasion to develop and present the fuller theory for the types of maintenance with other frequency ratios.



VI. The Magneto-Crystalline Properties of the Indian Braunites.

By K. Seshagiri Rao, B.A. (Hons.), Research Scholar
in the Association.

CONTENTS.

- SECTION I.—Introduction.
SECTION II.—Physical Characters of Specimens Examined.
SECTION III.—Experimental Methods and Results.
SECTION IV.—Chemical Composition of Braunite and Discussion of Results.
SECTION V.—Synopsis.

SECTION I.—INTRODUCTION.

Braunite is a natural ore of manganese which according to Dr. Leigh Fermor, who has made an extensive study of the available deposits,* is next to psilomelane the most important of the Indian manganese minerals and occurs in great abundance in various parts of the country. It is also found in other parts of the world both in crystallised and in massive form.† In spite of the enormous quantities of this mineral that are available it is only comparatively rarely that it is found in India in the form of measurable crystals. The detailed account of such occurrences will be found in the memoir already cited. A feature of this mineral which is not mentioned by Dana and to which attention has been drawn by Dr. Fermor is that it is invariably slightly magnetic.‡ In view of the great importance of the mineral, it was thought that a quantitative study of its magnetic properties in the various occurrences in India would be of interest.

* Memoirs of Geological Survey of India, Vol. XXXVII.

† Dana's 'System of Mineralogy,' 1894 Edn., page 233.

‡ Page 63 *loc. cit.*

SECTION II.—PHYSICAL CHARACTER OF SPECIMENS EXAMINED.

So far specimens from two localities have been examined by the author.

The first specimen, which was very kindly furnished for the experiments by Dr. Fermor, was taken from the collection of the Geological Survey of India (No. K 348) and was from Kacharwahi in the Nagpur District in the Central Provinces. The specimen as received showed on the exterior a patch of white attached material which was a felspar and a small amount of decomposed blanfordite brown in colour. Except for these impurities, however, the whole specimen was part of one crystal showing well-cut faces of a deep steel-gray colour showing sub-metallic lustre, and when the specimen was ground down on a carborundum wheel to the shape of a sphere 2·4 centimetres in diameter, it was found that there was no visible felspar or blanfordite left on the surface. The sphere showed at eight regions on its surface (forming roughly the corners of a cube) a brilliant metallic glitter due evidently to the crystalline cleavages. It seemed thus quite certain that the sphere thus obtained was part of a homogeneous crystal. The material was hard and rather heavy, the density being 4·8. The Kacharwahi crystals belong to the tetragonal system.

The other specimen was part of a hard lump of very pure ore picked up on the spot from the well-known hill at Kándri, in the Central Provinces, which contains probably the finest body of manganese ore yet found in India and certainly one of the finest in the world. This is a fine-grained crystalline ore composed of apparently of braunite with a certain proportion of admixed psilomelane. A piece was broken out of this lump and also ground into a sphere of 2·4 centimetres diameter. It showed no visible impurity of any kind. The ore was very friable apparently owing to the admixed psilomelane. Its density was 4·22.

SECTION III.—EXPERIMENTAL METHODS AND RESULTS.

The method of experimenting used was to grind the specimen into a sphere and determine the force pulling it into a non-uniform field, and from this force the average value of intensity of magnetisation for the sphere was calculated. If the susceptibility is constant throughout the spherical specimen, then to a first approximation the intensity of magnetisation calculated from the

pull is the value of intensity at the centre of the sphere. This average value was the value that was determined. The advantage of the method is that one can easily test the homogeneity of the material. This may be done by simply inverting the sphere with reference to the field, in which case the pull will remain unchanged only if the lower and the upper half are alike. Within the limit of accuracy of this test the specimen was homogeneous. The expression for the force exerted upon an inductively magnetised sphere along the y -axis is given by

$$F_y = \left[I_x \frac{\partial H_x}{\partial y} + I_y \frac{\partial H_y}{\partial y} + I_z \frac{\partial H_z}{\partial y} \right] v.$$

where

F = Mechanical force,

I = Intensity of magnetisation,

H = Magnetic field strength.

If the magnetisation lies in the xy -plane the last term drops out and if the direction of the x -axis be taken as that of the field at the centre of the sphere, the second term will have a very small value, so that the final simplified result becomes

$$F_y = I_x \frac{\partial H_x}{\partial y}.$$

To measure the force acting on the sphere, the following arrangement was adopted. A glass strip 2.5 cms. by 20 cms. was firmly clamped at one end to a rectangular brass piece fixed at right angles to a brass pillar moving up and down. The glass plate could be thus placed in proper position in the magnetic field. The other end of the plate carried a light wooden platform on which the sphere could be mounted. A pointer about 20 cms. long was attached to this end. In taking readings of the force the sphere was placed on the wooden platform. The bending of the glass plate as the field was slowly applied was measured by the motion of the end of the pointer which was observed through a microscope containing a scale in the eyepiece. The force acting on the sphere in dynes was calculated from the bending thus produced. For calibration the deflection produced by a known mass placed on the same position as the sphere was observed. With the arrangement used in the work a deflection of one division in the scale of the microscope eyepiece corresponded to about fifty dynes.

The magnet used in the work was a powerful electromagnet with large conical pole pieces which could be screwed into the iron cores of the magnet. These iron-cores moved in a groove in a strong iron bed plate and could be clamped in any desired position.

The field strength was determined by an exploring coil consisting of 40 turns and an equivalent area of 10.2 square centimetres. The coil was connected to a ballistic galvanometer in series with a suitable resistance. The galvanometer throw when the field was suddenly introduced was noted. The galvanometer was standardised by a condenser of capacity 0.2 microfarad.

The field gradient was determined by determining the fields at two points which were a definite distance apart.

Observations with the Specimen No. I.

The sphere was first hung at random in a magnetic field by a silk fibre. A directive twist was observed showing that the susceptibility of the crystal varied in different directions. Now when a crystalline sphere having different susceptibilities in different directions is placed in a uniform magnetic field, a restoring couple due to this difference and clearly proportional to it tends to twist the sphere so that the direction of maximum susceptibility coincides with the direction of the lines of force. This direction of maximum susceptibility was determined in this way:—

First a uniform magnetic field was obtained by replacing the conical pole pieces by flat ones. Two circles at right angles to each other were marked on the sphere. The sphere was hung at random in this field and turned in its own plane along these circles at regular intervals. The frequency of oscillation was observed at each position. The positions on the two circles at which it was maximum were marked. It was evident then that these were the positions at which the restoring couples were maximum since the frequency of oscillation increases with the magnitude of the couple. Since the restoring couples are proportional to differences of susceptibility, the planes perpendicular to the axes of rotation containing these two points would each clearly contain the axes of maximum susceptibility. Hence the intersection of these would give the direction of maximum susceptibility. If this were made the axes of rotation the direction of minimum would be easily obtained.

Tested in this way the sphere was found to possess a direction of minimum susceptibility. When it was mounted with this minimum as an axis of rotation the restoring couple became very small, showing the independence of magnetisation with direction in this zone. If the sphere were perfectly true it ought to rest easily in any position but as there were slight departures from complete sphericity and as the sphere could not be exactly placed in this position the couple did not absolutely vanish.

The direction of the maximum may be somewhat more quickly obtained by hanging the sphere mounted at random in a magnetic field since the direction of the maximum and that of the field should coincide. By repeating this process the plane of maximum susceptibility may be fixed.

The sphere was then mounted on the plate and placed in the non-uniform field produced by the conical pole pieces. Readings for the force were observed for different fields. Now with a constant susceptibility the force exerted must vary as $H \frac{\partial H}{\partial x}$, i.e. as H^2 . Within the accuracy of measurements made it was found that both in the direction of the minimum axis and in the perpendicular plane that F/H^2 was constant, i.e. the substance was paramagnetic. The maximum field used was about 4000 gauss.

Field and field gradient were measured by galvanometer deflections. Field gradient was found nearly proportional to the field, i.e. $H = K \frac{\partial H}{\partial x}$, and average value of K was used to calculate susceptibility. The value of susceptibility was thus found to be 0.4×10^{-3} .

Readings for the force in the direction of axis of minimum and to the plane perpendicular to it were nearly identical. The difference of the susceptibilities was found to be too small to be exactly determined by the method adopted. An idea of the magnitude of this quantity was obtained as follows:

The moment of the couple tending to turn the sphere in a magnetic about the axis of x from y towards z is given by

$$L = \frac{4}{3} \pi a^3 (\gamma B - \beta C).*$$

* Maxwell's Electricity and Magnetism, Vol. II.

When referred to principal axes of magnetism the expression becomes

$$L = \frac{4}{3}\pi a^3 \frac{r_2 - r_3}{(1 + \frac{4}{3}\pi\gamma_2)(1 + \frac{4}{3}\pi\gamma_3)} YZ$$

$$= \frac{4}{3}\pi a^3 \frac{r_2 - r_3}{(1 + \frac{4}{3}\pi\gamma)^3} F^2 \theta,$$

since when $X=0$, $Y=F \cos \theta$, $Z=F \sin \theta$, provided θ is small.

Thus by observing the time of oscillation when the sphere was mounted in a uniform field with its axis of rotation in the magnetic plane of symmetry and measuring the strength of the field, $\frac{r_2 - r_3}{\gamma}$, or the maximum difference of susceptibility was determined in terms of the average susceptibility. This difference was found to be about 2% of the value of average susceptibility.

The magnetic properties of the crystal as investigated thus bear a close resemblance to its physical characteristics. Braunite belongs to the tetragonal system, having a plane of symmetry perpendicular to the vertical axis. The value of $\frac{1}{2}$ the vertical axis is given by Dr. Fermor as * 0.99. Magnetically the crystal has been found to possess a plane of symmetry at right angles to which is the axis of minimum susceptibility, the difference of susceptibility being of the same order as that of the difference of crystallographic axes.

Specimen No. II.

As the specimen No. II was not a part of a single crystal but only a homogeneous mixture of braunite and psilomelane, the determination of its susceptibility only was made. The observations were made by comparison with those of specimen No. I. This was also found to be paramagnetic as in the case of No. I, but its susceptibility was only 75% of that of No. I.

It would be interesting in view of the interesting magnetic property possessed by Indian braunites, to make a detailed investigation of European braunites which as far as one could gather from the literature have not received any attention.

* Loc. cit. page 56.

SECTION IV.—CHEMICAL COMPOSITION.

Both of these specimens were analysed in the chemical laboratory of the Association. The results of these analyses are shown below :—

	Kacharwahi.	Kandri.
Iron	2.5%	2.6°
Manganese	54%	51.4
Silica and insoluble residue	11.8%	12.6

Both the specimens contain a relatively low proportion of iron.

Analysis of these specimens as recorded by Dr. Fermor are also given below for comparison.

	Kacharwahi.	Kandri.
Manganese	57.86	58.55
Iron	3.85	3.37
Density	4.79	4.28

It seems not unlikely that the magnetic property of braunite may be largely due to its manganese content, though on account of the presence of iron oxides in the substance it is difficult to pronounce decidedly on this point at present.

SECTION V.—SYNOPSIS.

The results of investigations upon the magnetic properties of Indian Braunites may be stated as follow :—

(1) Symmetry of braunite is magnetically that of an axis of symmetry, with a plane perpendicular thereto in which the magnetic susceptibility is independent of orientation.

(2) Along this axis of symmetry the susceptibility is minimum and in the plane it is maximum, the difference between the two being about 2%.

(3) Both in this plane and along the axis it is paramagnetic, the susceptibility being very low.

(4) All these magnetic properties bear a close resemblance to the physical characteristics of the crystal which are a crystallographic plane of symmetry at right angles to the vertical axis, the value of the axis $\frac{1}{2}$ being 0.99.

In conclusion, the writer wishes to thank Professor Raman who suggested the investigation and took much interest in its progress.



VII. Free and Forced Convection from Heated Cylinders in Air.

By Bidhubhusan Ray, M.Sc., Palit Research Scholar in the
University of Calcutta.

(Plate VI).

CONTENTS.

SECTION I.—Review of previous work on convection.

SECTION II.—Optical study of convection by the Foucault method.

SECTION III.—Electrical determination of temperature distribution in free convection.

SECTION IV.—Synopsis and discussion of results.

SECTION I.—REVIEW OF PREVIOUS WORK ON CONVECTION.

The phenomenon of convection from the surface of a heated body immersed in a fluid has received considerable attention both from the theoretical and the experimental points of view. If the cooling fluid be a gas, then, the variations of pressure, density, velocity, thermal conductivity and viscosity at different points, so complicate the problem that not only has little progress towards a complete mathematical solution yet been made, but for some time a true knowledge of its laws seemed nearly impossible. A. Oberbeck * gave the general differential equation for this problem, but being unable to solve it generally was satisfied with working out a solution for a special case. Later on, L. Lorentz † obtained a solution for the case of a vertically placed plane strip cooling in air. When the strip is protected from draughts, he found that

* Ann. Phy. 7, 271 (1879).

† Ann. Phy. 13, 582, 1881.

the heat convected away from the surface varies as $\theta^{5/4}$, where θ is the difference of temperature between the strip and air before it is heated by the strip, and this value agreed fairly well with some of the older experimental work. H. Wilson* attempted to solve the problem taking the viscosity into account but his solution is limited to a very special case.

In 1901 Boussinesq † took up the problem and first clearly stated the laws for the forced convection of heat for a body immersed in a stream of liquid when the flow is not turbulent. The same author after four years published a memoir containing a great number of successful solutions of the problem of the convection of heat from bodies of various shapes immersed in a stream of fluid. It is to be noticed here that according to Boussinesq's result, the heat loss due to convection of heat from bodies immersed in a stream of fluid varies as $V^{1/2}\theta$, where V is the velocity of the current and θ is the difference of temperature of the solid body above that of the surrounding fluid at a great distance. Kennelly, ‡ making elaborate measurements of the convection of heat from thin copper wires, remarked as follows:—"The lateral conduction through air is negligible because the air does not remain at rest but expands and flows convectionally. Consequently we may safely ignore conductive thermal loss. Convection loss from the air is a hydrodynamical phenomenon, involving the flow of air past the surface of the wire and the amount of heat which the moving stream can carry off. Very little seems to be known quantitatively about convection." Being unable to find any general solution, he was therefore satisfied with framing an empirical law to express his results. In the case of forced convection of heat his results agreed with those of Boussinesq for the cooling of cylindrical wires.

Russel, § following after Boussinesq, discussed the problem at some length. In order to solve the general differential equation and apply its results to bodies of various shapes and dimensions, Russell was compelled to make certain assumptions with regard to the nature of the fluid surrounding the solid. The liquid is

* Camb. Phil. Soc. Progs. 12, 406, 1902.

† Comptes. Rendus. Vol. 133, p. 257.

‡ Trans. Amer. Inst. E.E. 28, 263, 1909.

§ Phil. Mag. 20, 591, 1910.

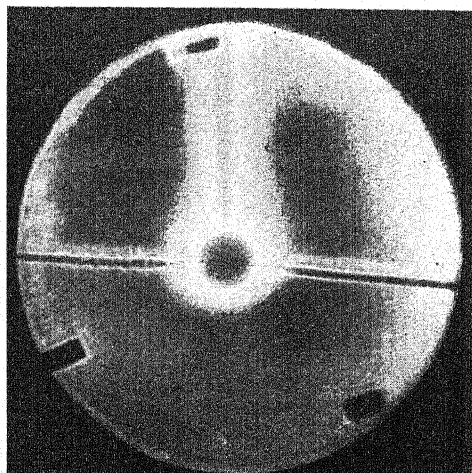


Fig. 1

Free Convection

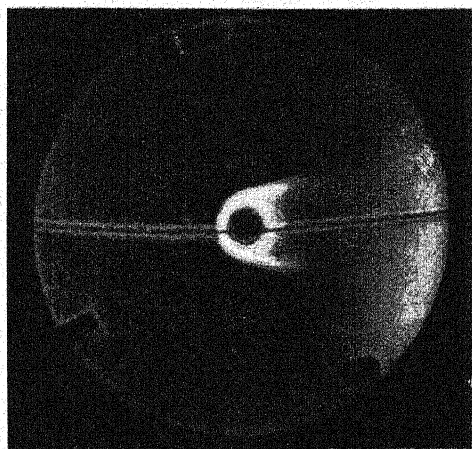


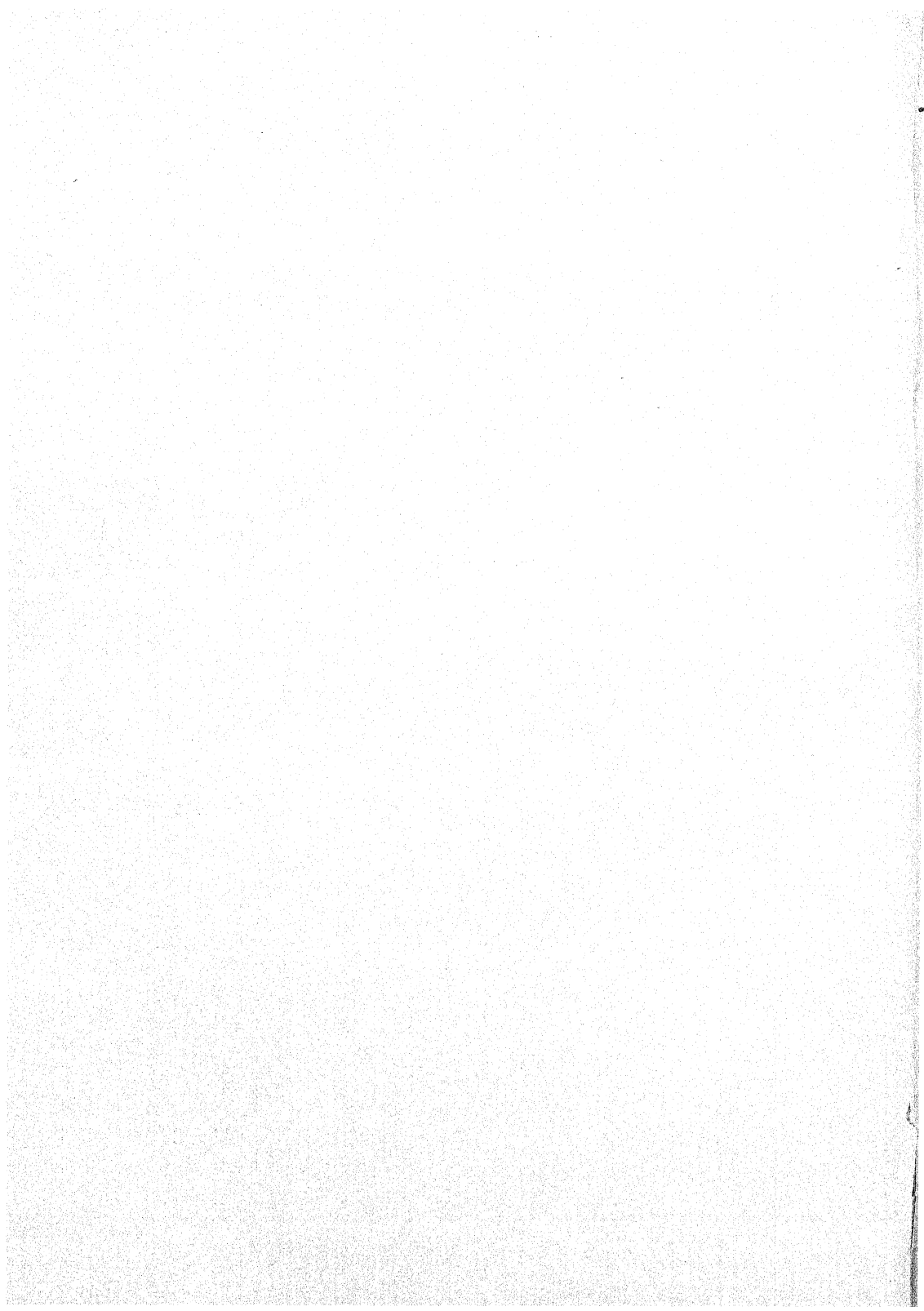
Fig. 2

Forced Convection



Fig. 3

Forced Convection



supposed to be opaque to heat rays, it has no viscosity and so it slips past the surface of the solid, also it is supposed to be incompressible. The further assumptions are, "The thermal conductivity of the liquid is very small and the variations in its density does not appreciably alter the trajectories of the liquid particles in the immediate neighbourhood of the solid from the shape they have during isothermal flow. The former assumption is true in most practical cases, and the latter is permissible when the velocity of the current is appreciable and no eddies are formed. The surface of the solid being cooled by the current is supposed to be isothermal, and the liquid in immediate contact with it at any instant is supposed to have the same temperature as the solid." In view of these very important assumptions we should expect the solutions to give only approximate values when applied to the problem of spheres and cylinders cooled by currents of air. The result, which Russell arrived at, and which is true for the two-dimensional flow round a solid of any shape immersed in a stream of liquid, agrees with that of Boussinesq; and when particular cases are considered it agrees with the work of Kennelly,* Ayrton and Kilgour† and other workers. All the investigators in calculating the heat loss from the surface of the body find it is proportional to the difference of temperature between the solid and liquid surrounding it. It is to be noted that Newton's law of cooling is thus verified when the cooling fluid is a liquid, and Newton himself enunciated his law with reference to the convection and not the radiation of heat. In several practical applications, Newton's law of cooling leads to results agreeing closely with experiment. In the theory of a hot-wire oscillograph Irwin‡ has assumed that the convection loss of heat from a heated strip immersed in oil is proportional to the difference of temperature between the metal and the oil. The very satisfactory coincidence between the theoretical and experimental result shows that the assumption is approximately correct.

Russel in the same paper shows that in the case of turbulent motion of water through a pipe, Newton's law is very approximately true. It is to be noted here, that we are now discussing

* Journ. Inst. Elect. Engin. 35, 364, 1905.

† Phil. Trans. Vol. 183, Pt. 1, p. 371, 1892.

‡ Journ. Inst. Elect. Engin. 39, 617, 1907.

the problem of forced convection of heat, but in the case of free convection Newton's law of cooling is not applicable in liquids, where Lorentz's law that the heat loss is proportional to $\theta^{5/4}$, where θ is the difference of temperature between the strip and air before it is heated by the strip is applicable.

I. Langmuir* differed from the previous workers in two of the important assumptions, namely in the effect of viscosity and the variation of the thermal conductivity due to temperature. This author contends that in the case of convection from small wires the effect of viscosity is most important. His own views are clearly given thus in his thesis on some reactions around glowing Nernst filaments: "According to the kinetic theory, the viscosity of gas increases with square root of the absolute temperature, the driving force of convection being proportional to the difference in density between the hot and cold gas increases only very slowly with increasing temperature. Therefore in the immediate neighbourhood of the filament the flow of gas is small and the heat must be carried away practically only by conduction. It is highly probable that at a very high temperature, the motion of a gas in the immediate neighbourhood of the wire would not perceptibly increase but probably decrease, while at the same time the heat conductivity of the gas would increase very greatly. Thus at a distance very near the surface of a body, heat conduction is more important than convection; so that heat will be carried only by radiation and conduction but not by convection." Several considerations had made it seem probable that the above theory would be fairly close to the truth. For instance it was found that the watts loss from a wire was the same, whether the wire was placed horizontally or vertically. Now the lines of flow of the heated air around the wire would be totally dissimilar in these two cases. Yet it was found that the energy necessary to keep a piece of pure platinum at a given temperature—the resistance being kept constant—never differed appreciably for horizontal or vertical wires, and at a bright red heat or above, the difference became negligibly small. This was a strong indication that the heat loss was dependent practically on heat conduction very close to the filament and that the convection current had practically no effect except to

* Phys. Rev. 34, p. 401, 1912.

carry off heat away after it has passed out through the film of adhering gas. The thickness of the film of gas through which conduction takes place can be calculated if the temperature of the wire, its diameter, and the heat conductivity of the gas are known. The last factor varies very much with temperature.

Langmuir, in calculating the heat loss from a heated wire, assumes that a thin cylindrical film of gas adheres to the wire through which heat travels only by conduction and radiation, and the thickness of the film of air is the distance heat must travel before the heat flux due to temperature difference becomes small compared with that due to convection. Throughout this film of air the viscosity, the thermal conductivity and specific heat of air varies greatly. Meyer * has given a formula for conductivity in terms of viscosity and specific heat (per gram) at constant volume, while Sutherland obtained an equation connecting viscosity and temperature containing two constants, which have been determined by Rankine.† M. Pier ‡ has obtained an equation for specific heat at constant volume in terms of temperature, and so connecting these formulae, Langmuir obtained a relation between conductivity and temperature, and applying the simple conduction formula in the thin film, he calculated the heat loss from the heated wire. His experimental result agrees fairly well with the theoretical formula.

Later on, the same author § published an account which is an extension of his earlier work to cover convection of heat from plane surfaces. Experiments are described in which the heat loss from plane surfaces at various temperatures is determined. After discussing the results of this experiment with his previous theory, he finds that the film theory is found not to apply to convection forced by air currents, but Russell's formula agrees well with the experimental data. Morris || has verified the application of the formula of the type obtained by Boussinesq to the cooling of fine wires heated by electric current to temperatures about 70°C above the surrounding air and for air velocities as high as 40 miles per hour.

Prof. L. V. King ¶ follows Boussinesq and transforms the

* Kinetic Theory of Gases. † Proc. Roy. Soc. Lon. A84, 181, 1910.

‡ Z. f. Electrochem. 15, 536, 1909 and 16, 899, 1910.

§ Amer. Electrochem. Soc. Trans. 23, p. 299, 1913.

|| Electrician, Oct. 4, 1912; also Engineer, September 27, 1912.

¶ Phil. Trans. A, Vol. 214, 1914.

general equation of conduction into a greatly simplified form in the case of a liquid flowing past a cylinder, reducing the problem to the case of a two-dimensional flow. The problem can now be transformed into a form having a known partial differential equation. The complete solution of this equation requires a knowledge of the conditions of heat transfer over the interface between the solid and the liquid. "The solution of this problem which gives results in best agreement with experiment for the case of convection of heat from small cylinders is that obtained by assuming the flux of heat to be constant over the boundary. As a result of high heat conductivity of the cylinder in the experiment carried out, the temperature of the cylinder may be considered constant over the boundary and there will therefore be a discontinuity in the temperature of the boundary; we assume that the temperature of the stream in contact with the cylinder becomes finally equal to that of the cylinder at the point where it leaves the boundary." The result he obtained for the loss of heat for small and large velocities of the air current agrees fairly well with experiment and with that of some of the previous workers in the case of forced convection. It is to be noted here that King based his theory on simple hydrodynamical flow in which there is slipping at the boundary between solid and liquid. Here he takes no account of viscosity, variation of thermal conductivity and specific heat near the surface. It is interesting to note that he gets the value of some constants which agrees fairly well with that of the other investigators working on different lines.

Kennelly* later on found, that for pressures between 5 and 3 atmos., the rate at which heat is dissipated varies not only as the square root of the speed of the wire as already found out but also approximately as the square root of the air pressure in accordance with the theoretical result of Boussinesq.

V. M. Schidlovski† in testing the truth of L. Lorentz's‡ formula for the case of cooling of a wire finds that the current sufficient to heat the vertical wire to redness leaves the horizontal one dull, and the effect due to conduction and convection is less in the case of vertical wire than when it was horizontal, and this

* Amer. Phil. Soc. Proc., Vol. 53, 55, 1914.

† Russian Physico-chem. Soc. J., 43, 132, 1914.

‡ Ann. Phys. 13, 582, 1881.

decrease is due to the ascending hot-air currents enveloping the wire.

T Barret *, while experimenting on radiation and convection from a heated wire in an enclosure of air, finds that as the temperature is raised the heat loss due to convection increases more rapidly than that due to radiation.

I. A. Hughes † has measured the loss of heat per c.m. length and also the loss at different velocities in the case of cooling of cylinders in a current of air, his results agreeing with those of the previous workers.

W. P. White ‡ has made an investigation to obtain data for the use in the design of calorimeters. According to him, leakage due to conduction and radiation conforms substantially to Newton's law of cooling. Heat transfer by convection, being due to air currents, whose temperature and velocity are both affected by thermal head, is more nearly proportional to the square of the temperature difference. The observed variation from Newton's law of cooling is to be ascribed to convection. He appears to have worked with cooling of bodies with small air gaps, so his result is not comparable with other workers who have worked with an infinite air gap. The results obtained both for eddyless motion and for turbulent flow are intended to serve as a practical guide in calorimeter designing.

In view of the fact that different (sometimes opposite) assumptions have been made by different investigators regarding the nature of the boundary conditions at the surface of the solid, and that there is much regarding the subject which is still obscure, the problem was taken up to study more fully

(1) the exact nature of the convection phenomena, and

(2) the temperature and velocity distribution round the heated body, and to find the law of flow both for free and forced convection of heat and to obtain material for dealing with the problem on a stricter hydrodynamical basis.

SECTION II.—STUDY OF CONVECTION BY THE FOUCALT METHOD.

Some idea as to the nature of the convection phenomena may be formed by using smoke or some such substance round the

* Lond. Phys. Soc. Proc. 28, p. 1, 1915.

† Phil. Mag. 33, 118, 1916.

‡ Phys. Rev. 10, 743, 1917.

heated body, and allowing strong light to fall on this smoke which becomes luminous and makes the nature of its movement visible to the eye. A much better method which has been used by the present author is an optical method of observation which is a modified form of the Foucault test or Töpler Schlieren apparatus. According to the simple laws of geometrical optics, rays issuing from a point can be focussed at another point if the optical instruments are perfect. An eye situated just behind the focus observes an illuminated field, but if a sharp-edged screen be gradually advanced in the focal plane, all light is suddenly cut off and the entire field becomes dark simultaneously. At this moment if there be any irregularity in the optical surfaces or in the medium through which the rays come to a focus, the rays are deviated from their proper course so as to escape the screening and the field becomes luminous. From the picture of the luminous patch the nature of the irregularity of the optical appliances or of the medium through which the rays are passing can be studied.

The arrangement of the experiment is as follows: A copper cylinder of radius 7.2 millimeters is used for the experiment. A heating coil made of nickel-chromium wire, wound on a frame of mica, similar to that used in platinum-resistance thermometers, is inserted into the cylinder, which is kept in a horizontal position by means of two stout wires as seen in the photograph. A current of 1.8 amp. is passed through the coil, and the cylinder which was insulated is thereby heated.

White light from a small circular hole, illuminated by the electric arc, is incident upon a good achromatic lens, which forms a sharp image of the hole at a considerable distance from the lens. A thin circular metal disc fixed on a thin sheet of mica is put in the focal plane, and the whole arrangement is so adjusted that the metal disk just completely cuts off the image of the circular hole formed by the achromatic lens. A telescope is placed with its object-glass just behind the metal disk attached to the sheet of mica and is pointed towards the lens. The copper cylinder from which the convection is observed is placed immediately in front of the lens with its axis along the optical axis of the system, and the observing telescope is focussed on the end of the cylinder.

When the cylinder is heated by passing the electric current,

the field of view of the telescope which is previously just moderately dark soon begins to show the luminous streams of air rising from the cylinder. The phenomenon seen gives a vivid visual picture of the nature of the temperature distribution round the cylinder and of the motion of the air.

Figure 1 in Plate VI illustrates *free* convection from the heated cylinder. The luminous patch seen is very broad and diffuse, and quite steady. The horns pointing upwards show the path along which the heated air rises. It will be seen that some distance above the cylinder and between the two horns we have a distinct darker vertical strip in which the temperature appears to be less than within the horns.

Figure 2 in Plate VI shows the effect observed when a gentle current of air from an electric fan is forced horizontally at right-angles to the cylinder. The horns of the luminous area are here much better defined, and it was noticed that they were somewhat unsteady in position owing to the fluctuations in the current of air. As visually seen the thin tips were much more pointed than would appear from the photograph. The photograph was taken with quite a short exposure. The luminous patch here is much sharper and was visually observed to be far more brilliant than in the case of free convection, showing that the gradient of temperature is greatly increased by the forced draught.

Figure 3 in Plate VI also shows a case of forced convection, the velocity of the draught being greater here than in the case of Fig. 2. The features mentioned above, are here seen in more accentuated form. A considerable difference in the thickness, brightness and sharpness of the luminous fringe in front of and behind the cylinder will be noticed.

In the photograph reproduced in the Plate, one of the luminous horns is seen to be distinctly brighter than the other. This is due to the fact that the circular metal disk which cut off the image of the source was not exactly symmetrically placed. It was very interesting to watch the luminous streamers on the ground glass of the camera when the position of the disk was slightly altered or when irregular draughts of air were allowed to fall on the cylinder. The adjustment of the disk to the final position of symmetry had to be made by actual trial. In studying the free convection, it was found necessary to protect the cylinder

from irregular draughts by surrounding it by a large wooden box.

Observations were then made of the diameter of the luminous fringe surrounding the cylinder and of the velocity of the wind. It was found that as the velocity of the wind was increased the diameter of the patch decreases at first very rapidly but after a time very slowly until finally with the increase of the velocity of air, it is difficult to detect any change in the width of the luminous patch. The following measurements of the diameter of the patch give an idea of the effects observed :—

Velocity of the wind.	Angle width of patch in the field of the observing telescope.
Free convection	$0^{\circ}-14'-44''$
5.6 ft. per min.	$0^{\circ}-12'-30''$
8.5 ft. per min.	$0^{\circ}-11'-28''$
9.8 ft. per min.	$0^{\circ}-9'-43''$
12.5 ft. per min.	$0^{\circ}-8'-47''$
18.2 ft. per min.	$0^{\circ}-8'-10''$

SECTION III.—ELECTRICAL DETERMINATION OF TEMPERATURE DISTRIBUTION ROUND THE CYLINDER IN FREE CONVECTION.

To confirm the general indications furnished by optical observation, the distribution of temperature in the region surrounding the cylinder was studied by an electrical method. A thermocouple of fine copper and constantan wires was used, the junction between them being made by electrical fusion. Care was taken to see that the junction of the two wires occupied a very small region. One end of the couple was placed in ice and the other placed near the cylinder. The position of this end with respect to the distance and direction from the cylinder could be determined by means of a telescope having a millimeter scale in the eyepiece. The thermocouple was balanced against a calibrated potentiometer bridge wire

and the balance point carefully noted. The cylinder was protected by a wooden box enclosing it, and very great care had to be exercised to avoid disturbing currents of air.

Five sets of readings for each position of the couple were taken and the mean of the five was used for drawing the form of the isothermals round the cylinder.

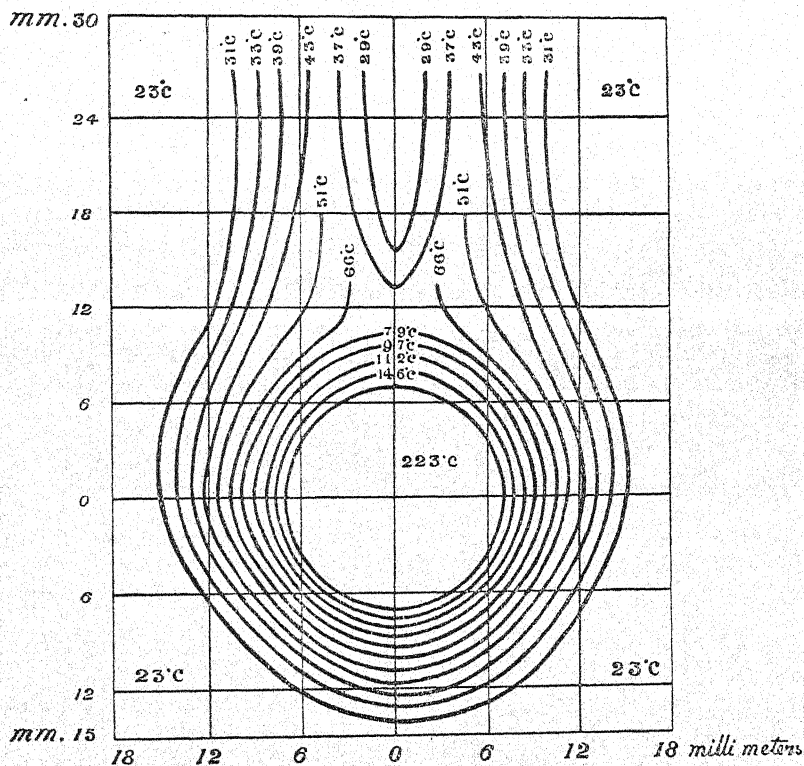


FIG. 4.—Temperature Distribution around Heated Cylinder.

The curve (Fig. 4) shows a striking agreement between the temperature distribution round the heated cylinder as observed electrically and as indicated by optical observation and illustrated in Fig. 1 in Plate vi.

It is to be noticed that the junction of copper-constantan occupies a finite space and hence when it is placed near the cylinder, the flow of heat round the cylinder is disturbed by its very presence. Some heat will be carried along the wires away

from the junction and some heat will be radiated away from the cylinder to the junction. These effects cannot be entirely eliminated but may be diminished by making the junction occupy a very small space and having the wires of the couple made very thin.

One important point to be noticed is that the isothermals are crowded together on the lower side of the cylinder, and diverge on the upper side of the cylinder.

SECTION IV.—SYNOPSIS AND DISCUSSION OF RESULTS.

The present paper describes the results of an experimental study of the free and forced convection of heat from horizontally held cylinders in air. The general character of the temperature distribution round the cylinder was optically observed by the method of the Foucault test and quantitatively determined by thermo-electric measurement of the temperature of the air. The following are the principal features observed :

1. In *free* convection, there is a relatively large region of heated air surrounding the cylinder, the temperature gradients are relatively small, and the region above the middle of the cylinder is actually cooler than the region vertically above its two edges. Immediately on the surface of the cylinder itself, the gas appears to be quiescent, the movement increasing as we pass away from the cylinder on either side. There is probably no actual discontinuity of temperature at the boundary itself, but there is a rapid fall of temperature as we move away from the surface.

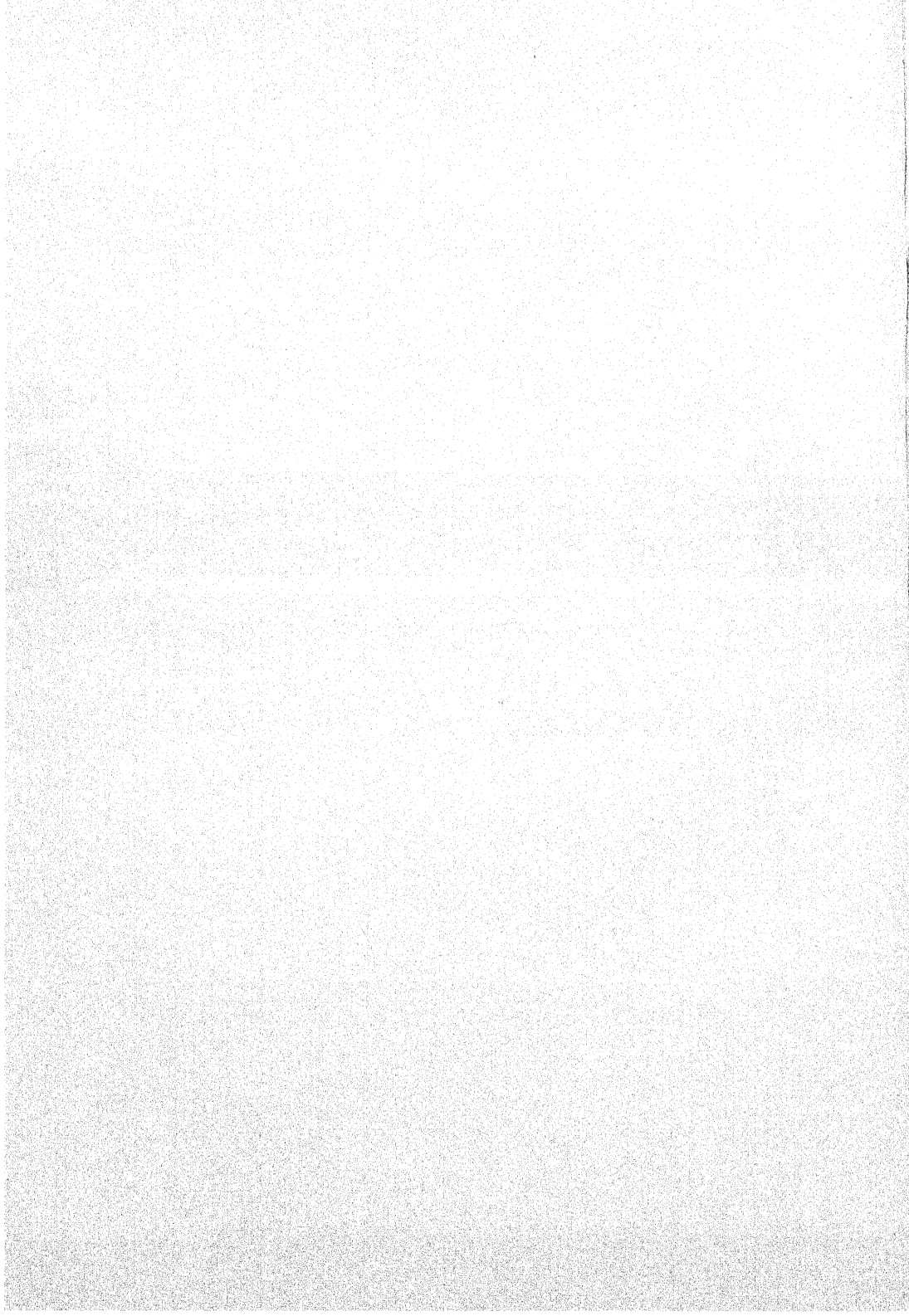
2. In *forced* convection, the layer of heated air is much thinner, and the temperature gradients are accordingly greater. There is a noticeable difference in the thickness of the heated fringe of heated air on the leeward and windward sides of the cylinder, and this thickness decreases with increasing velocity of the air current. The 'horns' or currents of heated air moving away from the cylinder are much sharper in forced convection than in free convection.

3. The optical observations and the experimentally determined form of the isothermals in the fluid surrounding the cylinder suggest the question as to how far the hypothesis of King that there is a discontinuity of temperature between solid and the surrounding gas represents actual fact. The evidence suggests

rather than in the case studied by the present author that there is little or no slipping between the fluid and the solid at the boundary, and no actual discontinuity of temperature though there is a rapid fall of temperature close to the boundary. Though there is no actually stationary film of *finite* thickness of gas in contact with the solid, there appears to be very little motion in the immediate neighbourhood of the boundary, and presumably at the boundary itself the gas is at rest. By considering the effect of viscosity of the fluid, it would seem possible to explain the effects observed. The mathematical theory of convection contained in King's paper appears to require modification taking the effect of viscosity into account to enable the observed form of the isothermals to be explained, and the agreement obtained between King's theory and the observed convection-losses of heat from small cylinders may require to be interpreted afresh.

The experimental work of finding the distribution of temperature round the cylinder was carried out in the Palit Laboratory of Physics, and the author offers his best thanks to Prof. C. V. Raman, Palit Professor of Physics, for suggesting the problem and for his lively interest in the progress of the investigation.

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92, Upper Circular Rd., Calcutta,
1920.*



VIII. Experiments on Impact.

By A. Venkatasubbaraman, B.A.

In a paper which has appeared in the Physical Review for April 1920, Prof. C. V. Raman has shown how the well-known theory of impact developed mathematically by Hertz may be extended and applied to the problem of the transverse impact of a sphere on a plane elastic plate of finite thickness. The theory involves the consideration of the proportion of the kinetic energy of the impinging body transformed into energy of wave-motion in the elastic plate. Starting with the assumption that, to a first approximation, the duration of impact is the same as on Hertz's theory for the case of an elastic plate of infinite thickness, the potential and kinetic energies carried off by the flexural waves generated in the plate by impact are calculated on certain simplifying assumptions regarding the form of these waves. The result finally obtained is that, e , the coefficient of restitution of the impinging body is given by the formula

$$e = \frac{f\rho a^2 - 0.56 M}{f\rho a^2 + 0.56 M}.$$

where

$$a^2 = \pi T f \sqrt{E/3\rho} (1 - \sigma^2)$$

T is the duration of impact,

E is Young's Modulus for the plate,

σ is Poisson's ratio for the plate,

$2f$ is the thickness of the plate,

ρ is the density of the material of the plate,

and M is the mass of impinging body. If T (the duration of impact) be taken to have the value given by Hertz's theory for a plate of infinite thickness, we may expect a good agreement between the formula and the observed coefficient of restitution for

moderately thick plates. For *thin* plates, it would be more nearly correct to take T as the actual duration of impact which would no doubt differ somewhat from that for the case of an infinitely thick plate.

To test the foregoing formula, the author has made a series of observations using a set of polished hard steel balls of various diameters and a set of glass plates of different thicknesses. These materials were chosen as, provided the size of the impinging sphere is not too great and the velocity of impact is not too large, no permanent deformation of the impinging bodies results from the impact. For velocities of impact greater than a certain limit depending on the size of the impinging sphere, the collision results in the formation of percussion-figures of beautiful geometrical form which have been observed and described by Professor C. V. Raman ('Nature,' Oct. 1919). If a percussion-figure be formed, we cannot naturally expect the formula for the coefficient of restitution to remain valid, and it is accordingly necessary to restrict the comparison with it to cases in which the spheres and the velocities of impact are moderately small.

The accompanying table gives the results for the coefficient of restitution obtained by the writer for a certain moderate velocity of impact. The coefficient of restitution calculated from the formula are also shown, T being taken in the formula to be the duration of impact as given by Hertz's theory for an infinitely thick plate.

The following facts emerge on an examination of the figures shown in the Table of results. For the thicker plates, the experimental values for e are smaller by two or three per cent. than the theoretical values. This is evidently due to various minor causes of dissipation of energy not being taken into account in the theoretical treatment. For moderately thick plates, the calculated and observed coefficients of restitution agree well. Theory and experiment also agree in the case of very thin plates in giving a zero coefficient of restitution. In other words, in such cases, the sphere on impact with the plate remains in contact with it and behaves nearly in the same way as a perfectly inelastic body. But in certain intermediate cases where the thickness of the plate is less than about half the diameter of the impinging sphere but not so small as to give a zero coefficient of restitution, the observed

COEFFICIENT OF RESTITUTION.


Velocity of Impact 234 cms. per second.

Thickness of Plate in Centimetres.	Diameter of Spheres in Centimetres.											
	0.791		0.714		0.637		0.555		0.396		0.314	
	Calc.	Obsd.	Calc.	Obsd.	Calc.	Obsd.	Calc.	Obsd.	Calc.	Obsd.	Calc.	Obsd.
2.53	0.98	0.95	0.99	0.96	0.99	0.97	0.99	0.97	0.99	0.98	0.99	0.98
1.93	0.97	0.95	0.98	0.96	0.98	0.96	0.99	0.97	0.99	0.97	0.99	0.98
1.29	0.93	0.91	0.94	0.93	0.96	0.93	0.97	0.94	0.98	0.95	0.99	0.98
1.01	0.89	0.87	0.91	0.89	0.93	0.91	0.95	0.93	0.97	0.94	0.98	0.97
0.71	0.79	0.77	0.83	0.80	0.86	0.83	0.89	0.86	0.94	0.90	0.96	0.96
0.49	0.61	0.60	0.67	0.66	0.73	0.71	0.79	0.77	0.89	0.86	0.93	0.94
0.35	0.36	0.38	0.44	0.45	0.53	0.53	0.62	0.61	0.79	0.77	0.86	0.89
0.225	0	0	0.03	0.13	0.15	0.23	0.28	0.34	0.55	0.55	0.69	0.79
0.150	0	0	0	0	0	0	0	0.12	0.21	0.28	0.42	0.63
0.105	0	0	0	0	0	0	0	0	0	0.08	0.09	0.43

values of e are somewhat larger than the calculated values. This appears to indicate that in the case of such plates, the actual duration of impact is somewhat greater than that given by Hertz's theory for the case of an infinitely thick plate, and that better agreement between theory and experiment would be obtained if such actual duration was taken as the value of T in the formula. (Of course, this would only be permissible provided the actual duration is finite and not very much greater than given by Hertz's formula.)

Some observations made by the writer show that when the velocity of impact on the plate is so great as to result in the formation of percussion-figures, the coefficient of restitution falls considerably below the value given by the formula above. This is what may be expected in view of the fact that the production of the percussion figures must result in dissipation of energy which involves a decrease of the coefficient of restitution. In this connection, it is also worthy of note that much higher falls and therefore larger velocities of impact may be used in the case of small spheres than in the case of larger spheres without resulting in production of percussion-figures. As the magnitude of the stresses per unit area set up by impact is (according to Hertz's theory) the same for the large as well as for small spheres, this observation suggests that the duration of impact is distinctly a factor in determining whether or not the impact should result in the production of a percussion figure. The author has also observed that the area of the percussion-figure increases appreciably with increasing velocities of impact, and also, as might be expected, with increasing size of the impinging sphere. From the observations of the decrease of the coefficient of restitution, it is possible to calculate the energy required for the production of the percussion-figure and to find how it varies with the area of the internal fracture which constitutes the percussion-figure.

The author has also found that appreciable deformations from planeness of the surface of the glass plate result from the production of percussion-figures and may be detected by optical methods. This phenomenon is no doubt intimately connected with the question of residual strains in the plate left after production of the internal fracture.



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Vol. VI, Parts III & IV.

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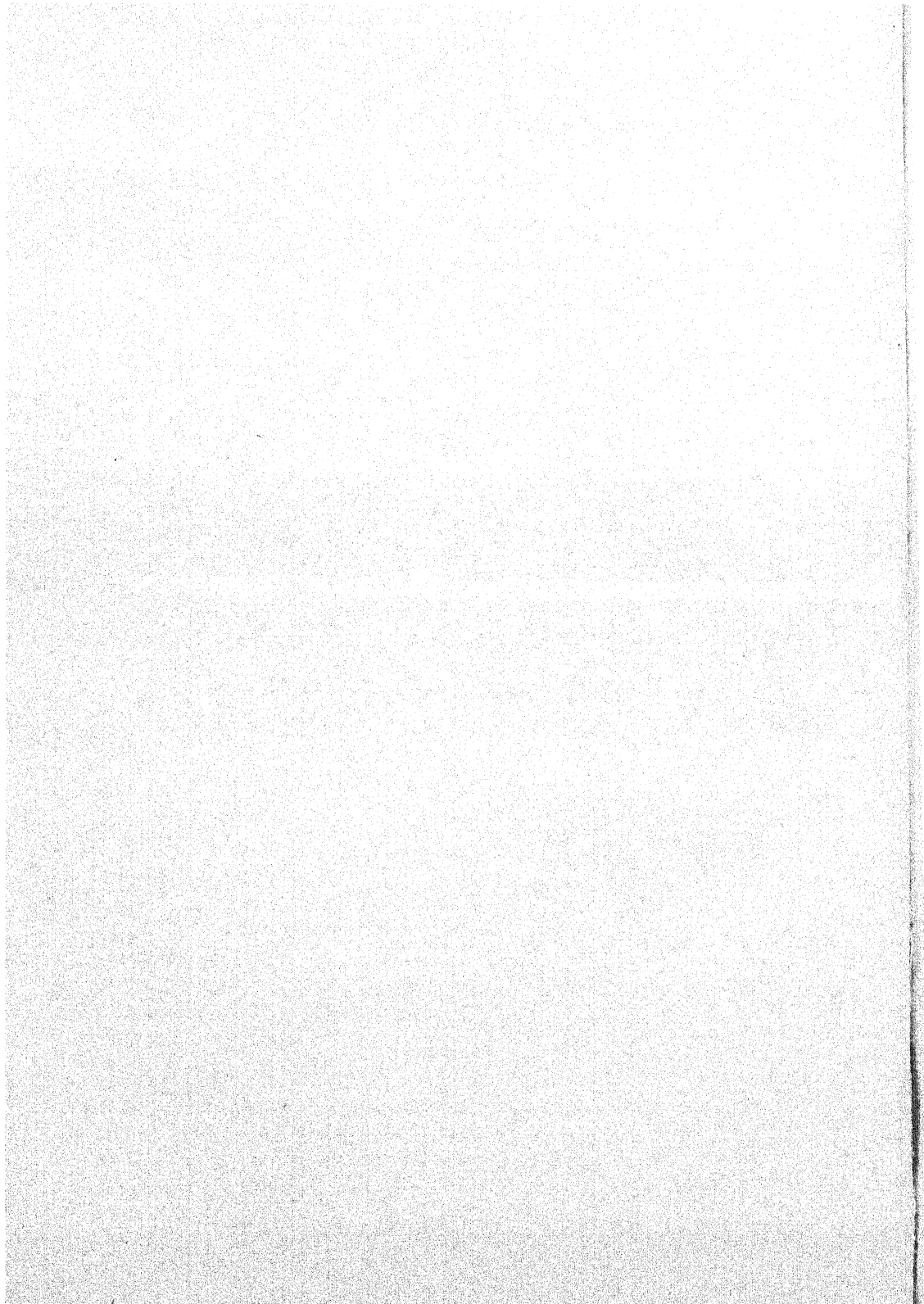
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IX. The Theory of the Flute.

By Dr. G. T. Walker, C.S.I., Sc.D., F.R.S., Director-General
of Observatories in India.

Among the more attractive applications of mathematical physics are those which have been made to musical instruments; and although in most cases the main principles have been truly laid down there are details still unexplained which may be of some theoretical or practical interest. Some recent experience in designing piccolos of different bores has brought out various points in which the existing theory of the flute appears incomplete, and I propose to give a short account of these in the hope that it may stimulate a further theoretical examination of the subject. An excellent account of the many papers dealing with the acoustics of pipes will be found in Winkelmann's *Handbuch der Physik* (Vol. II, pp. 446, 7. Leipzig, 1909).

2. The questions at issue fall naturally into four classes as they are concerned with (a) the shape of the bore of the instrument; (b) the position of the holes and open end; (c) the size of the holes; (d) the method of blowing; and (e) the method of fingering. Considering first the bore, Boehm pointed out that while a flute with a large internal diameter possessed facility and tone in the lowest notes the highest notes were unmanageable; and $\frac{3}{4}$ of an inch or 19 mm. is generally accepted as the standard for the ordinary *D* flute whose length is about 24 inches. For the ordinary *D* piccolo a bore of $\frac{3}{8}$ " gives a rather feeble lowest octave and the usual bore of $\frac{7}{16}$ " is better: one of $\frac{1}{2}$ " is brilliant in its lowest notes, but rather difficult in the highest. Boehm also pointed out that a flute with a purely cylindrical bore cannot be in tune throughout its range, the notes of the second octave being flat by comparison with those of the first and those of the third by comparison with those of the second. It would thus

appear that the correction for the open end or for a large hole in a cylindrical tube increases appreciably with the frequency of the note.¹ Boehm rightly gave as the remedy for this difficulty a constriction of the bore of the upper portion of the tube (that containing the mouth-hole) so that the bore diminishes gradually from its full dimensions at a quarter of the length from the top end to nine-tenths of the dimensions at the mouth-hole. The same result is apparently produced in a conical flute by making the top third cylindrical and the lower two-thirds conical, the diameter at the open end being about $\cdot 7$ of that of the cylindrical portion.

3. The length of the tube to produce a given note will differ from the half of the corresponding wave length by the sum of the corrections for the mouth-end of the flute and for the open end. Similarly the positions of any of the holes will depend on the sum of the corrections for the corresponding hole and for the mouth-end. An obvious way of determining the facts is to measure the distances of a series of holes of equal size from the mouth-hole in an actual flute. When the unknown but constant correction has been subtracted these distances must bear to the unknown distance for the fundamental note a series of known ratios depending on the musical intervals of the notes in question. We thus have a number of equations with only two unknowns, these being the constant correction (the sum of the corrections for the mouth end and for a hole of the given size) and the distance for the fundamental note. In this way I have deduced from my flute, with a bore of $\frac{3}{4}$ ", that the distances of the $\frac{1}{2}$ " holes from the mouth-hole are less by $2\cdot40$ " than their position if there were no corrections; for a $\frac{5}{8}$ " hole the corresponding amount is $2\cdot27$ ". Also the length from the mouth-hole to the open end is less by $2\cdot0$ ". Now when the bore is $1/30$ of the wave length Bosanquet (see page 425 of the volume of Winkelmann above referred to) found a correction for the open end of $\cdot543$ of the radius; and for a bore of $1/12$ of the wave-length he found $\cdot635$. For the lowest note C of my flute the bore is $1/34$ of the wave-length, and I estimate that the correction for the open end should be $\cdot53$ of the radius. Thus I estimate $\cdot20$ "

¹ This is in agreement with Bosanquet's observation, referred to in para. 3 below, that the correction for the open end of a short tube of a given bore is greater than for a long tube.

as the correction for the open end of my flute. The correction for the mouth end is thus 2'0" diminished by 0'2", or 1'8"; hence that for a half inch hole is 2'4 - 1'8 = 0'6"; and that for a $\frac{5}{8}$ " hole is '47. This is for a tube of thickness $\frac{1}{8}$ "; for a greater thickness the correction would have to be increased. When designing an instrument of another internal diameter, say $\frac{3}{8}$ ", the correction for a proportional hole of $\frac{1}{4}$ " will obviously be $0'6" \times \frac{1}{2} = 0'3"$; for a $\frac{5}{16}$ " hole it will be 0'24". It is however convenient to make holes of various sizes; and for holes greater in diameter than '6 of the bore the correction can be got by extrapolating from those given. But for holes smaller than '6 of the bore other considerations must be taken into account.

4. The obvious plan for a concert flute with a bore of $\frac{3}{4}$ " would be to make all the holes of the same size, say $\frac{1}{8}$ ", not only for the notes rising in semitones from *D* to *C*-sharp of the bottom octave, but also for the bottom *C* and *C*-sharp and also for the auxiliary topmost holes of the *D* and *D*-sharp shake-keys. But if this were done the effect of pulling out the tuning slide to flatten the general pitch would have a much greater effect when the topmost holes were speaking than when the bottom holes were speaking, and the instrument would be out of tune with itself. This defect is greatly diminished by making the topmost holes smaller than the rest; in fact as a hole is made smaller its effect seems to depend more on its size and less on its position. In practice the upper *C*-sharp hole, which is perfectly in tune in the lowest octave, is in a position which would, if the hole were uniform in size with the lower holes, correspond much more nearly to *D* than to *C*-sharp. Also the *D* and *D*-sharp holes, which, like the *C*-sharp, are only about $\frac{11}{32}$ in diameter, are in positions closely corresponding to the wave-lengths of *D*-sharp and *E* respectively: thus in the fingering of *B*-flat in the third octave the so-called *D* key is used, as lying nearly in the place of *E*-flat. This same key is used for the fingering of *D*-sharp in the fourth octave, while the *D*-sharp key is opened in the usual fingering for *E* in the fourth octave.

5. To a beginner in the flute playing the most obvious fact in blowing a low note is that as the force of blowing is increased, keeping the lips quite still, the pitch rises slightly, and on reaching a certain amount of force the note will jump up an octave, then becoming usually rather flat in pitch. If the force is further

increased the next harmonic may be reached but it also will be flat in pitch. After some skill has been acquired the player will learn to keep the pitch of a note constant when blowing with varying degrees of force by pushing out his lips when blowing softly and drawing them in and blowing more downwards when blowing with more force. His production of the harmonics will then depend on the position and shape of his lips and each harmonic can be produced either loudly or softly. The control on the pitch of a note that can be exercised by altering the lips is much greater when the topmost hole left open by the fingers is one of those near the mouth-hole than when it is near the bottom end.

6. Let us now consider the question of fingering, or, in other words, the selection of the holes which are to be covered in order to produce the successive notes. For an ordinary concert flute in *D* no question arises over the bottom octave or the middle octave as far as *A* : all that is necessary is to have the 'speaking' hole open and one or two holes below it for 'ventilation,' in order to produce a free tone. When we are wanting the octave above any note, say *B*-flat, we must produce a note which instead of having 'nodes' merely at the mouth-hole and at the *B*-flat hole, has also one half-way between (corrections being ignored). But for such a wave-length there will also be a node at a point whose distance from the mouth-hole is $3/2$ times that of the *B*-flat hole. This point corresponds to a note a fifth deeper than *B*-flat. Thus we may produce the note we want by opening merely the *B*-flat hole and the *E*-flat and lower holes: and it is easily verified that while with the ordinary fingering it is difficult to blow the note up to pitch when a very gentle sound is wanted, with this theoretical fingering the gentlest note will not be flat.¹ Naturally with certain patterns of flutes this method is not applicable to all notes. For example, a *B* natural in the middle octave cannot be sounded in this way on an ordinary Boehm flute; for covering the *F* hole will close the hole through which the *B* natural speaks. For the *C* and *C*-sharp in the middle octave we may define the vibrations in the tube still further by a third node,

¹ Fingerings of this type are recommended for pianissimo notes in one of the foreign hand-books.

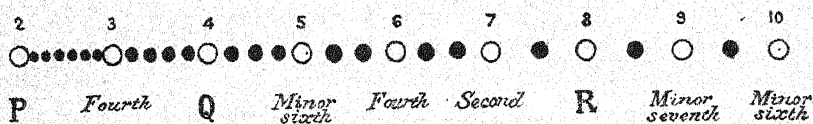
for the loop at $4/2$ times the distance from the mouth-hole is the bottom *C* or *C-sharp* key. For the latter note we should have merely three holes open, the upper *C-sharp* hole, the *F-sharp* and the bottom *C-sharp*. In practice however there is no advantage over the theoretical method previously described.

7. For notes in the third octave the methods of cross-fingering to be found in the text-books differ in several cases from the theoretical fingering owing to the mechanical connections between certain of the keys. For the theoretical fingering of any note, say *E-flat*, we remember that the distance from the mouth-hole to the first node is to be a quarter of that from the mouth-hole to that through which the lowest *E-flat* speaks; and we must, as far as possible, open the holes corresponding to all the nodes—that is at $1/4, 2/4, 3/4, 4/4, 5/4, 6/4, 7/4$, etc., of this distance. The corresponding frequencies are 4, 2, $4/3$, 1, $4/5$, $2/3$, $4/7$, etc. For *E-flat* in the third octave, for example, the ordinary fingering uses the harmonics corresponding to $4/3$ and 1; the *E-flat* key is open and that a fourth above this, i.e. the *A-flat* key. There is no key with frequency 2 on the ordinary Boehm flute, for the *E-flat* shake-key is so small that its position is above that really corresponding to *E-flat* for a high note, and if this is opened in addition to $4/3$ and 1 the resulting note is too sharp. If however the *D* shake-key is opened instead a good note is obtained. For *F-sharp* the holes to be opened are those producing *B*, *F-sharp* and *D*; and the note so produced is true if the holes are in their true places. But if the *D-sharp* key is, as usual, kept pressed down for the sake of rapidity of execution, the *F-sharp* note becomes slightly flat; and in order to counteract this error some makers put the *F-sharp* hole in a flute at a trifle higher than its true place. The error in the bottom octaves is very slight and is not so serious as an error in the top octave where the flute is most conspicuous in an orchestra. If the hole for *F-sharp* is not placed above its theoretical position the top *F-sharp* will be rather flat unless the *D-sharp* key is left closed. For *G-sharp* the frequencies $4/3$, 1, $4/5$, $2/3$ correspond to a fourth above, *G-sharp*, a third below and a fifth below: or to *C-sharp*, *G-sharp*, *E*, and bottom *C-sharp*. This fingering gives a ringing note that is a trifle sharp; but it is too complicated for ordinary use, and may be simplified by opening all the holes below the *E*. We may further simplify

by opening the *C* natural hole near the top of the tube and thus only close the *E* and *F* holes in addition to the ordinary fingering on the Boehm flute. This will be useful if the ordinary fingering gives a flat note. Similarly for the high *B*-flat we open the holes that produce *B*-flat, *G*-flat, and *E*-flat, leaving the holes below this last open. This is rather more complicated than the ordinary fingering but is easily played up to pitch while with the ordinary fingering flatness is hard to avoid, especially when playing softly. The theoretical *C*-sharp in the topmost octave (with top *C*-sharp, *A*, *F*-sharp and bottom *C*-sharp keys only open) is a trifle sharp. But if the top *C*-sharp key be closed the note is in tune. It is however much more complicated than the ordinary fingering, and is of no use except when playing pianissimo.

8. For the few notes of the fourth octave that are of any importance the distances of the holes opened from the mouth-hole

Figure I



must be proportional to $4/8, 5/8, 6/8, 7/8, 8/8, 9/8, 10/8$, etc., and the frequencies to $2, 8/5, 4/3, 8/7, 1, 8/9, 4/5$, etc. For *D* these give *D*, *B*-flat, *G*, *E* (roughly), *D*, *C*, and the ordinary fingering for a Boehm flute uses the holes *C*-sharp (because the *D* shake-key is too highly placed), *B*-flat, *E* and *C*. The other holes are not available owing to the mechanical relationships between the keys, and perhaps on this account the note is not appreciably sharp. For the *E* of this octave the holes to open are those of top *E*, *C*, *A*, *F*-sharp (roughly) *E*, *D*, *C*. The ordinary Boehm fingering uses the *D*-sharp shake-key, *A*, *F*-sharp, *E*, *C*, all other holes being closed. In this way the note is difficult to sound; but if the theoretical fingering is followed, the *C* hole is to be opened in addition and the lower *D* hole may also be open, i.e. the so-called *C*-sharp and *C*-natural keys should not be closed. The sounding of the *E* then becomes fairly easy; in fact it is easier than the top *E*-flat or *D*. If it is found a trifle sharp the *D*-sharp shake key

Figure II

The keys over the holes for B and G are dotted as in a Boehm flute they are not controlled by the fingers directly, but through other keys.

D [#]	○	●	●	●	●	●	●	○	○	●
D	●	●	●	●	●	○	●	●	●	●
C [#]	●	●	●	○	○	●	●	○	●	○
C	○	○	○	○	○	○	○	○	○	○
B	⊖	⊖	○	⊖	⊖	⊖	○	⊖	⊖	⊖
B ^b	●	○	○	●	●	○	○	●	●	●
A	○	●	●	●	○	●	○	○	○	●
G [#]	○	○	○	○	○	○	○	○	○	○
G	⊖	⊖	⊖	⊖	⊖	⊖	⊖	⊖	⊖	⊖
F [#]	●	●	○	●	●	○	○	○	○	○
F	●	○	●	●	○	●	○	●	●	●
E	○	●	●	○	●	●	●	●	○	○
D [#]	○	○	○	○	○	○	○	○	○	○
D	●	●	○	●	○	○	○	●	○	○
C [#]	●	○	○	○	○	○	○	○	○	○
Note	E	F	F [#]	G [#]	A	1 2 B ^b		C [#]	E	F [#]

The fingerings of E and F are solely of theoretical interest. For F[#] and A closing the D[#] hole keeps the note from being flat if there is any tendency in that direction. The G[#] fingering may be simplified by opening the lower D and D[#] holes. For B^b the first fingering is strictly theoretical and is inclined to be sharp. So it is better to close the upper D key as in the second fingering. For C[#] there is no difficulty in closing the D key while opening the D[#], but either of these alone will suffice.

may be closed. In the same way the theoretical *F*-sharp, with the *C*-sharp, *G*-sharp, *F*-sharp, *E*, *D* holes open (not the *B* since opening this involves opening the *B*-flat also) may be sounded ; but it is too difficult to be of practical value.

9. Some of these fingerings are illustrated in the diagrams which are appended. Figure 1 is a general scheme for the theoretical fingering of any note above the first octave. In it are indicated a series of holes a semitone apart of which the open circles represent open holes and the black circles closed : the holes extend over a greater range than an ordinary flute contains, but only those actually on the flute for the note in question need be considered. The numbers immediately above the line are proportional to the distances of the holes in question from the mouth-hole ; so that from 2 to 4 is an octave and from 4 to 8 is an octave. Let us now suppose that we want the note which is an octave above that corresponding to *P*, or two octaves above that corresponding to *Q*, or three octaves above *R*. The open circles will then obviously indicate the holes to be opened, and their intervals above the fundamental sounded note will be those given immediately below. If, for example, we want the top *B* which is two octaves above the *B* in the lowest octave, the hole marked 4 must be the *B* hole ; the fourth above is *E*, so the *E* hole above this is to be opened if there be one. The hole marked 5 is a minor sixth above *B*, or *G* ; the hole marked 6 is a fourth above or *E* ; that marked 7 is a second or *C* sharp ; that marked 8 is *B* which is beyond the range of the ordinary flute. On a Boehm flute the hole for 4 is the *D*-sharp shake-key, while the closing of the holes between 5 and 6 will close the hole for *G*, so all holes below the *G* hole are left open : the note produced is sufficiently controlled by the five holes from 4 to 5, and the opening of all holes below the *G* practically cancels the lower part of the tube.

10. In Figure 2 are given some fingerings of high notes on a Boehm flute, indicated by theory where these differ from the standard to be found in the ordinary text-books. The notes in question are *E*, *F*, *F*-sharp, *G*-sharp, *A*, *B*-flat and *C*-sharp in the third octave, with *E* and *F* sharp in the fourth octave.

X. On Wave-Propagation in Optically Heterogeneous Media, and the Phenomena observed in Christiansen's Experiment.

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Physics in the Benares Hindu University.

(Plate VII.)

CONTENTS.

- I.—Introduction.
- II.—The Transmitted Light.
- III.—The Colours of the Halo.
- IV.—The Surface Colours.
- V.—Experiments with Liquids.
- VI.—The Colours of Doubly Refracting Powders.
- VII.—Summary and Conclusion.

I.—INTRODUCTION.

The very striking and instructive optical experiment known under his name was first described by Christiansen in 1884.¹ He found that glass or other isotropic transparent solid in the state of fine powder which is ordinarily white and quite opaque, becomes transparent in respect of a limited region of the spectrum on immersion in a mixture of carbon disulphide and benzol in proportions suitable to make its refractive index nearly equal to that of the powder. The mixture exhibits a beautiful display of colour in the transmitted light and in the halo with which bright objects appear to be surrounded when viewed through it. A general description of the effects observed will be found in Prof. R. W. Wood's *Treatise on Physical Optics*.² It would appear,

¹ *Annalen der Physik* XXIII, p. 298.

² Second edition, p. 110.

however, that in spite of the very attractive features of the experiment, the phenomena exhibited by these mixtures have not yet been thoroughly investigated. In two important papers, the late Lord Rayleigh¹ drew attention to Christiansen's experiment, especially to one remarkable feature observed in it, namely, the very small range of wave-lengths transmitted by these mixtures, and showed how this feature might reasonably be explained on the principles of the wave theory. Apart from this, however, there are a number of other important questions which arise and as yet remain to be answered. What is the exact distribution of intensity of light among the various wave-lengths transmitted? How does this distribution depend on the size of the particles in the powder and the length of path of light through the mixture? Has the halo a definite structure, or must we be content with vaguely describing it as having a colour complementary to that of the transmitted light? How is the halo influenced by the factors mentioned above and the other circumstances of the experiment? It is proposed in the present paper to examine these and other questions relating to the propagation of light in optically heterogeneous mixtures. It is believed that the inquiry may be of interest, especially in view of the very wide application in recent years of the method of immersion in the determination of refractive indices and dispersive powers.

II.—THE TRANSMITTED LIGHT.

According to the principles of geometrical optics, a mixture of two isotropic media should regularly transmit only the rays for which the refractive indices of its components are identical, and should scatter all others in various directions. Actually however, in Christiansen's experiment, we find that this is not the case and that part of the incident light is regularly transmitted and is capable of giving well-defined optical images even when the refractive indices differ by an appreciable quantity. For instance, if a flat-sided cell containing the mixture be placed between the collimator and the prism of a spectroscope of which the slit is illuminated by sunlight, we find that a finite region of the spectrum continues to be visible and is in sharp focus, the Fraunhofer lines

¹ Scientific Papers, Vol. II, p. 433 and Vol. IV, p. 392.

being quite clearly distinguishable in every part of it. The range of wave-lengths thus regularly transmitted in the spectrum varies with the fineness of the particles and the thickness of the layer of powder. Christiansen in particular found that the transmitted region increased in width with the fineness of the powder used and indeed comprised nearly the whole visible spectrum when the observations were made with the finest powders of all separated by prolonged elutriation. The explanation of these effects is obviously of considerable interest.

The late Lord Rayleigh¹ proposed a treatment on the following lines. He first determined the probability that the number of particles of one component which a ray will encounter during its passage through the mixture, will differ from the mean number m by less than $\pm r$, and showed that this will be sufficiently great and equal to .84 for a value of $r = \sqrt{2m}$. This gives a measure of the phase differences likely to arise in the passage of the light through the heterogeneous medium, from which it is inferred that the range of wave-lengths transmitted may be expected to be $\frac{m}{\sqrt{2m}}$ times that which is not resolved by a prism of equal thickness with a dispersive power equal to the difference between the dispersions of the two media. The expression for the latter resolving power is well known to be

$$\delta\lambda = \frac{\lambda}{t \frac{d\mu}{d\lambda}}$$

and even if m is taken to be equal to $\frac{t}{d}$ —a value evidently greater than the real one—the total range of wave-lengths transmitted by the powder should be

$$\begin{aligned} \Delta\lambda &= \sqrt{\frac{t}{2d}} \cdot \frac{\lambda}{t \frac{d\mu}{d\lambda}} \\ &= \frac{1}{\sqrt{2td}} \cdot \frac{\lambda}{\frac{d\mu}{d\lambda}} \dots\dots\dots (1) \end{aligned}$$

¹ Scientific Papers, Vol. IV, p. 395.

where t is the thickness of the layer of the medium and d is the average size of the particles in it.

It would appear, however, from the investigation made by the present writer that the observable width of the region of transmission in the spectrum depends to a large extent on the intensity of light used and in all the measurements made, it was found to be much greater than that indicated by Rayleigh's formula. Even Christiansen's estimate is much greater than what equation (1) will give for his data. The fact is that the distribution of intensity among the wave-lengths transmitted is of the exponential form, being maximum for $\mu - \mu' = 0$ and diminishing more or less rapidly to zero for large values of $\mu - \mu'$ (μ and μ' are the refractive indices of the two components of the mixture). We cannot, therefore, strictly speaking, talk of a region of transmission without at the same time defining the limiting intensity of the light at the ends of that region.

Fortunately, considerations of the wave-theory of light combined with the theory of probability readily give us an expression for the intensity of the transmitted light. We may treat this problem on much the same lines as adopted in a recent paper by Mr. Chinmayanandam¹ in discussing the specular reflection from a rough surface. Consider the configuration of the wave-front immediately on emergence from the heterogeneous medium. It will not be a perfectly plane wave-front, because different portions of it will have been unequally retarded by the random distribution of the particles of the powder. We may assume that the deviations of the wave-front from perfect planeness follow the well-known law of errors. Certain parts of the wave-front would have traversed more of the solid and less of the liquid and *vice versa*. Let x denote the difference between the average thickness of the solid traversed by the whole wave-front and the thickness traversed by a specified element of the wave-front. Then the total area of the portions of the wave-front in advance of or behind the mean, for which this difference lies between x and $x+dx$, may be taken proportional to $e^{-Ax^2}dx$. The resultant vibration due to these portions of the wave-front is, therefore,

¹ Physical Review, Vol. XIII, Feb. 1919, p. 96.

$$y = e^{-Ax^2} \cos \left\{ wt - \frac{2\pi}{\lambda} (\mu - \mu')x \right\} dx,$$

if the equation of the vibration for the mean wave-front be $y = \cos wt$. The average vibration due to a unit area of the complete wave-front will then be for the wave-length λ ,

$$y = \frac{\int_{-\infty}^{+\infty} e^{-Ax^2} \cos \left\{ wt - \frac{2\pi}{\lambda} (\mu - \mu')x \right\} dx}{\int_{-\infty}^{+\infty} e^{-Ax^2} dx}.$$

This will make the average intensity

$$I = e^{-\frac{2\pi^2}{A} \left(\frac{\mu - \mu'}{\lambda} \right)^2} = e^{-B \left(\frac{\mu - \mu'}{\lambda} \right)^2} \dots \dots \dots (2)$$

where the constant B will depend on the size of the particles d and the thickness of the layer of powder t and possibly to some extent on the arrangement of the particles.

To test the foregoing formula and to find experimentally the manner in which the constant B depends on d and t , a number of observations were made by the writer. Powdered glass was the substance used being the most convenient and suitable for the purpose. By passing it in succession through several sieves of wire gauze and cloth having a different number of meshes to the inch, powders of varying grades of fineness were obtained, the particles in each of which lay between definite and determinable limits of size. The samples of powder thus obtained were then thoroughly cleaned, washed and dried. For the purpose of the observations, the powder was placed in a cell with parallel faces of optically good glass, and enough carbon disulphide was poured into the vessel. Benzene was then slowly added till the refractive index μ of the glass and μ' of the liquid mixture became equal for red light and this began to be freely transmitted. Further additions of benzene shifted the region of transmission towards the violet end of the spectrum, thus affording opportunity for making observations at any desired stage. The thickness t of the layer through which the light had to pass was varied by inserting into the powder pieces of clean glass plate of different thicknesses.

The constant B was determined by placing the cell containing the powder between the collimator and the prism of a wave-length spectroscopie and determining the range of wave-lengths transmitted, then screening the light by a screen which transmitted a known fraction k of the light falling on it, and then again determining the diminished range of transmission. The source of light must be quite steady and a 500 c.p. half-watt lamp was used. It is clear that if the limit to which the eye of a particular observer in particular circumstances can just see, corresponds to a certain absolute value of intensity, the value of the intensity at the limit in the second case after screening must be equal to that at the limit in the first case. So that

$$ke^{-B\left(\frac{\mu-\mu'}{\lambda}\right)_2^2} = e^{-B\left(\frac{\mu-\mu'}{\lambda}\right)_1^2}$$

and

$$B = -\frac{\log_e k}{\left(\frac{\mu-\mu'}{\lambda}\right)_1^2 - \left(\frac{\mu-\mu'}{\lambda}\right)_2^2} \dots \dots \dots (3)$$

For comparison three such screens were employed for which the values of k as determined by a rotating sector photometer were

$\frac{1}{8.41}$, $\frac{1}{13.44}$ and $\frac{1}{22.5}$ respectively, and the values of B determined

by each of them separately agreed well. The variation of B with t and d is set forth in tables I and II.

TABLE I.
 $d = .0055$ c.m.

t in c.m.	B	$\frac{B}{t}$
1.1	.0528	.0480
.78	.0315	.0404
.45	.0161	.0358
.30	.0130	.0433
.12	.00705	.0588

TABLE II.

$t = .78$ c.m.

d in c.m.	B	$\frac{B}{d}$
.0055	.0315	5.69
.0142	.0515	3.63
.0245	.0900	3.67
.0410	.206	5.03

In view of the unavoidable uncertainty in the observations and changes in the temperature of the mixture and also the non-constant nature of the arrangement of the particles, the values of $\frac{B}{t}$ and $\frac{B}{d}$ may be seen to be fairly constant for the large variations of the values of t and d . So that B must be taken to be of the form

$$B = ctd.$$

and
$$I = e^{-cta \left(\frac{\mu - \mu'}{\lambda} \right)^2} \dots \dots \dots (4)$$

where c is a constant which may depend on the arrangement of the particles but is otherwise an absolute one. It may be noted that this formula makes the dimensions of c as of a pure number.

The result $B = ctd$ may also be deduced theoretically if we combine (1) and (2). If we remember that

$$\frac{1}{2} \Delta \lambda \frac{d\mu}{d\lambda} = (\mu - \mu')$$

at the limit of visibility, we get

$$\begin{aligned} \frac{\mu - \mu'}{\lambda} &= \frac{1}{2} \frac{\Delta \lambda}{\lambda} \cdot \frac{d\mu}{d\lambda} = \frac{1}{\sqrt{8td}} \\ \therefore \left(\frac{\mu - \mu'}{\lambda} \right)^2 &= \frac{1}{8td} \dots \dots \dots (5) \end{aligned}$$

Now, if we suppose that at the limit given by Rayleigh's formula, the intensity falls down to a fixed value, we must have from (2)

$$B \left(\frac{\mu - \mu'}{\lambda} \right)^2 = \text{constant}$$

from which (5) will give

$$\frac{B}{8td} = \text{constant}$$

or

$$B = ctd.$$

The average value of the absolute constant c as deduced from the experiments described above is about 7.

From the expressions given above, it is a simple matter to calculate the proportion of the energy that appears in the transmitted light, given the size of the particles, the thickness of the layer and the dispersive powers of the media or *vice versa*. It would be interesting to continue the work and test experimentally whether the formula given by the foregoing theory continues to be valid in the case of powders much finer than those used by the author in his observations and which transmit a much larger portion of the spectrum. No quantitative estimate is given by Christiansen of the size of the particles in the case of the finest powders used in his observations which according to him transmit nearly the whole of the visible spectrum. Otherwise this question could very easily have been tested. It seems not unlikely that the opalescent coloured precipitates of potassium fluosilicate experimented upon by Wood may be usefully employed for further work on this point.

III.—THE COLOURS OF THE HALO.

The energy which fails to appear in the transmitted light goes mostly into the halo seen surrounding the light-source, the proportion lost by reflection or absorption being small except possibly in the case of large thicknesses. Some observations have been made by the writer of the manner in which the scattered light is distributed in the halo. A strong source of light—an arc-lamp or a 1000 c.p. half watt lamp—was placed behind a small hole which was observed through the mixture of the powder and the liquids from a distance of about two meters. Naturally, the source appeared to have the colour of the transmitted light and was surrounded by a halo which was also coloured and which extended to very large angles. Hitherto the halo has been described as

having a colour complementary to that of the transmitted light. This suggests that the halo is all of one colour. But the most casual observation was enough to show that this was not the case and that the colour of the halo changes progressively from the centre outwards. The part of the halo immediately surrounding the source is of nearly the same tint as the source itself and shows a fine mottled or granular structure similar to that exhibited by diffraction halos in monochromatic light which has been discussed by De Haas¹ in a recent paper. The following table shows the colours of the different parts of the halo in various cases, the order being from the centre outwards:—

TABLE III.

TRANSMISSION.	HALO.
Red	Red, Orange, Yellow, Greenish-yellow, Green, Blue, Indigo.
Yellow	Yellow, Orange, Orange-yellow, Yellow, Yellowish-green, Green, Greenish-blue, Blue, Indigo.
Green	Green, Orange-yellow, Rose, Lilac.
Blue	Blue, Greenish-yellow, Orange-yellow, Rose, Red.
Indigo	Indigo, Blue, Bluish-white, Greenish-white, Yellowish-white, Reddish-white.
Violet	Violet, Blue, Bluish-white, Greenish-white, Yellowish-white, Reddish-white.

The explanation of the variation of the colour of the halo in its different parts becomes evident on examining the spectrum of the light scattered by the powder. In order to do this, all that is necessary is to take a direct-vision spectroscope and point it towards the cell containing the mixture. But the light proceeding in directions other than the one desired must be prevented from falling on the slit of the spectroscope. This can be ensured if the distance between the powder and the spectroscope slit is sufficiently

¹ K. Akad. Amsterdam Proc. XX, pp. 1278-1288, 1918.

great (over a meter or two) and the area of the surface of the powder sending out the light to the slit is itself considerably restricted by a narrow aperture. Specially interesting are the effects observed at very small angles. Fig. 1 in the plate shows the spectrum in the direction of the freely transmitted light and consists of a very narrow band for which $\mu - \mu'$ does not appreciably differ from zero. Figs. 2-5 in the plate are the spectra at gradually increasing obliquities. Fig. 6 is the complete spectrum reproduced for comparison of the widths of the bands obtained in the various cases. From these it is clear that proceeding from the centre, the halo consists at first mainly of two very narrow bands in the spectrum on either side of the region of transmission; these bands gradually widen out and also become more and more separated as the obliquity of observation is increased. The actual colour of the halo in any case is the resultant due to the mixture of the two bands in the spectrum. At a sufficiently great obliquity, these bands will together comprise almost the whole of the spectrum except for the wide region included between them in which the intensity is deficient or actually zero. In this case the colour of the halo may be said to be roughly the complementary of that of the transmitted light—roughly because a considerable part of the spectrum is missing from it over and above the portion regularly transmitted through the powder.

Further information regarding the halo is obtained by using a monochromatic light-source—say the green light of the mercury-vapour lamp—and varying $\mu - \mu'$ for this light by altering the proportions of carbon disulphide and benzene in the mixture. But a better and more convenient method is to keep the mixture unaltered and change the wave-length of the light used. A monochromator must be used for this purpose. It is most instructive to observe, as the screw of the instrument is slowly turned, the halo decrease in size, vanish altogether—the intensity of the regularly transmitted light being at this instant a maximum—and then increase once again. From the general appearance of the monochromatic halo and some photometric observations made by the author, it would appear that the distribution of light in it is at least approximately of the form e^{-ax^2} , the illumination being maximum at the centre and gradually diminishing symmetrically to zero. The constant a must also decrease with increasing $\mu - \mu'$ and

SPECTRA

Fig 1



Transmission

Fig. 2



Scattering

Fig. 3



Ditto
at greater obliquity

Fig. 4



.. ..

Fig. 5



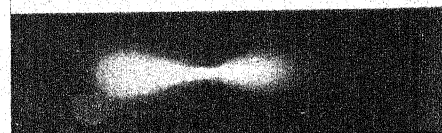
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Fig. 6



Complete Spectrum

Fig 8



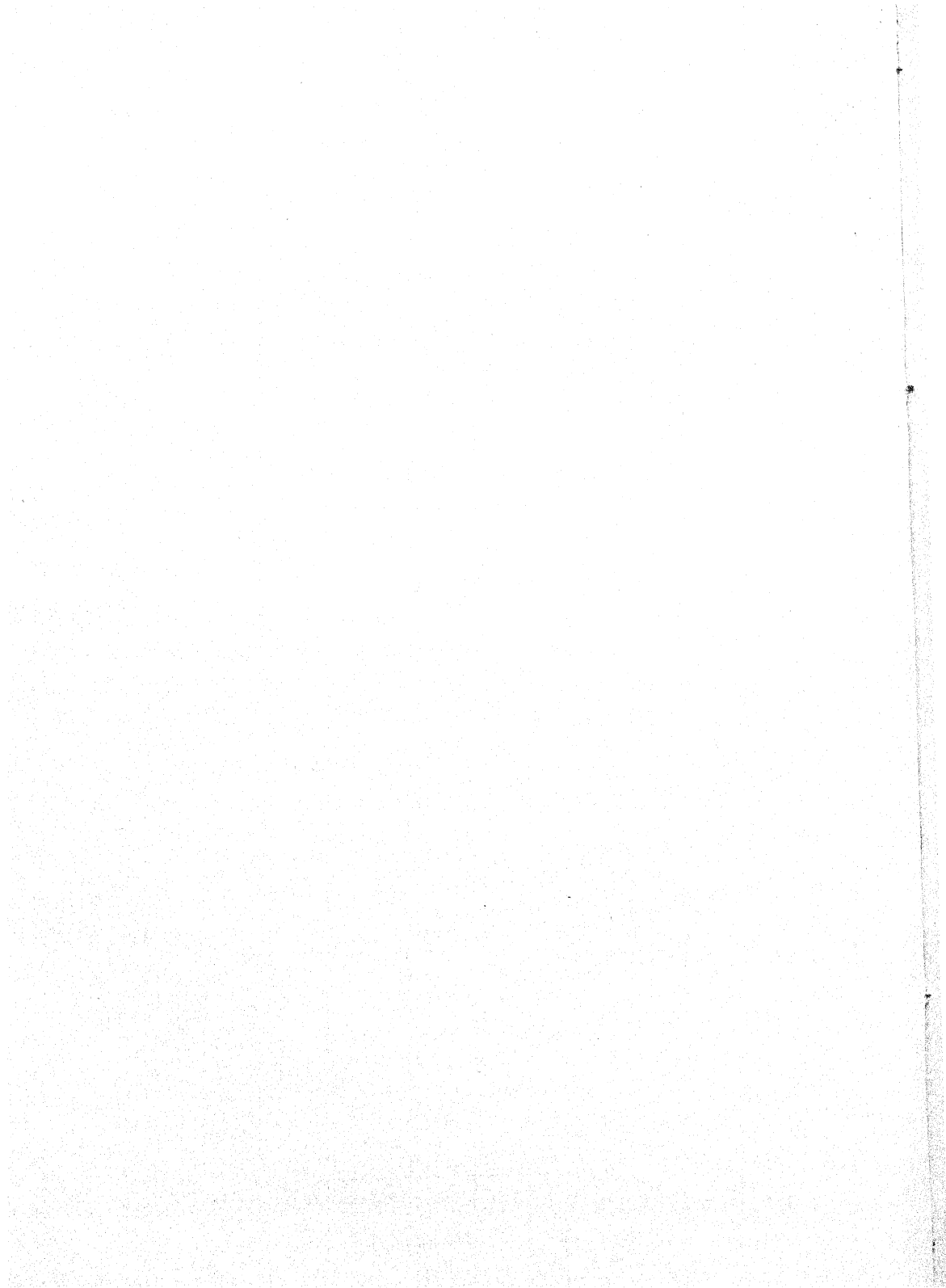
Scattering by
Glass

Fig. 9



Ditto
Quartz

Phenomena observed in Christiansen's Experiment.



very rapidly too; for the extent of the halo, as far as the eye can see it, is very small at first and increases very rapidly as $\mu - \mu'$ increases. It is also clear that it should be an even function of $\mu - \mu'$ so that at least as a first approximation a may be taken to be of the form $\frac{b}{(\mu - \mu')^2}$, where b depends on the size of the particles d , the thickness of the layer of powder t and possibly on the wave-length λ .

In fig. 7 are drawn a few curves of the form $y = e^{-ax^2}$ for different values of a and correspond to the illumination curves for different wave-lengths. The abscissae represent the distances from

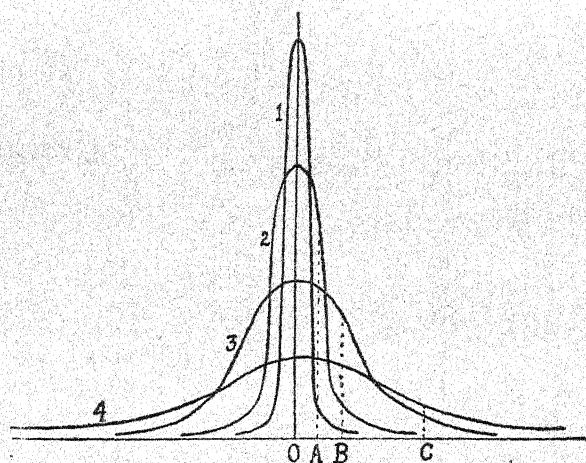


FIG. 7.

the centre of the halo. Curves 1, 2, 3, 4, correspond to cases in which the difference $\mu - \mu'$ has steadily increasing values. It is at once apparent from the figure that at the point A of the halo, λ_1 corresponding to curve 1 is practically absent and the principal illumination is due to wave-lengths near about λ_2 and λ_2' (one greater and the other smaller than λ_1) corresponding to curve 2. Similarly at B , even λ_2 and λ_2' are almost absent and the main illumination is due to wave-lengths in the neighbourhood of λ_3 and λ_3' . This completely agrees with the photographs in figs. 2-5 and satisfactorily explains why the dark band in the centre and the two bright bands on either side gradually widen out as we proceed further and further out from the centre of the halo.

The writer is not yet in a position to elucidate the exact manner in which the numerical value of the constant b determining the angular size of the halo depends on d , t and λ , or to offer a detailed theory. The general nature of the variations was, however, evident from the observations, namely that the halo increases in size when the particles in the mixture are smaller or the thickness of the layer is larger. Much more photometric work is necessary before this question and the manner in which λ comes in can be definitely settled. Reference may, however, be made here to the explanation of the granular structure of the central part of the halo which has already been mentioned. This is evidently connected with the arbitrary variations in the phase at different points of the wave-front emerging from the mixture. Any quantitative theory of the distribution of light in the halo can give only the statistical or average value of the intensity in the neighbourhood of any given direction. At individual points of the halo, large deviations from this average value are inevitable. Near the centre of the halo, as we have already seen, the scattered light is confined to a very small region of the spectrum, and hence the arbitrary fluctuations of intensity are specially marked. Further out from the centre of the halo, the effects of different parts are superposed and the irregular fluctuations of intensity are less easily noticeable.

That the size of the halo surrounding the source varies with the difference $\mu - \mu'$ of the refractive indices and hence is different for different colours of the spectrum is very clearly illustrated by fig. 8 in the plate. This is a picture obtained by interposing the cell containing a uniform layer of the mixture in the path of the rays forming a focussed spectrum on the photographic plate, the distance between the plate and the cell being about half a centimeter. Owing to the scattering action of the layer, the different parts of the spectrum are spread out to different extents, this spreading being evanescent for the point at which $\mu = \mu'$. There is a distinct asymmetry in the photograph which would seem to indicate that the size of the halo is greater on the large wave-length side and less on the other.

IV.—THE SURFACE COLOURS.

A very striking effect which does not appear to have been

previously described is the brilliant colour of the boundary between the glass powder in the lower part of the vessel and the liquid above it. This colour is different from the rest of the powder and is ever so much more brilliant. In order to observe it well, it is necessary that the surface of the powder should be level, which can be easily secured by gently tapping the vessel, and the line of sight should be almost tangential to this surface. Another very remarkable thing about this boundary is that it does not show the same colour on both sides, so that when regarded from the lower or the glass side, it shines with one colour which changes into quite a different one when the observation is made from the upper or the liquid side. Both these appearances are to some extent summarised for the various stages in the following table in which the order of the colours is for gradually increasing angles with respect to the transmitted light, the most beautiful stage being when the yellow is freely transmitted.

TABLE IV.

TRANSMISSION.	LIQUID SIDE.	POWDER SIDE.
Red.	Green, Blue.	Deep Red.
Yellow.	Green, Blue, Violet.	Orange, Red.
Green.	Blue, Violet.	Yellow, Orange, Red.
Blue.	Violet.	Green, Yellow.
Violet.	Faint Violet.	Green, Yellow.

The reason for these colours appears to be that the surface of separation acts more or less like a totally reflecting barrier, those wave-lengths for which $\mu > \mu'$ being totally reflected on the glass side, while those for which $\mu < \mu'$ appear on the liquid side.

The fact that the boundary colours appear brilliant when the line of sight is almost tangential to the surface is easily explained if we remember that the difference of refractive indices is at most very small and hence the angle of incidence should not differ much from a right angle.

These colour effects are identical with coarse as well as fine powders, the only difference being that with fine powders the boundary line appears much sharper and more well defined.

V.—EXPERIMENTS WITH LIQUIDS.

The colours obtained when equal volumes of glycerine and turpentine are shaken together have been noticed previously and Wood remarks that they are of a similar nature to those observed in Christiansen's experiment. It was thought that it would be worth while to examine this case more closely. At room temperatures, pure glycerine has a refractive index slightly less than that of spirits of turpentine throughout the visible spectrum, the difference being greatest at the violet end and least at the red. The effects noticed at ordinary temperatures are not therefore particularly striking. The author has found however that at higher temperatures very lively colours may be obtained. The liquids may be put together in a small flat-sided flask which is corked and then warmed up by being put in a beaker containing boiling water for a few minutes. On taking it out and shaking up the mixture and looking through it at a source of light, we find no transmission at first but merely a halo with violet centre surrounded by the other spectrum colours in regular order. As the mixture cools, violet, blue, green, yellow and red light is transmitted in succession and the colour of the halo undergoes corresponding changes. Finally, the transmitted light vanishes, and we have only a halo with a yellowish white centre and a blue-violet margin. The size of the halo in all these cases is much smaller than in the case of the glass powder and consequently it is much easier to distinguish the colour of the different parts of the halo. Another interesting point about this halo is that the mere shaking up of the mixture increases its size as one should naturally expect from the greatly diminished size of the drops.

The upper liquid, viz. turpentine, becomes very quickly free of the large drops of glycerine which fall down rather rapidly, but very fine drops remain floating in it for a considerable time. When an object is viewed through this apparently clear liquid, a halo is seen to surround it, while the object itself remains practically white. This halo is quite different from the halo referred to above and which is seen through the lower liquid where one can see many

large drops. It is much smaller in size and does not show much gradation of colour. Generally only one or two tints are visible. When the transmission is near the red end of the spectrum, this halo consists of blue and violet colours only. When the transmission is towards the other end, the halo is brown and red. In the intermediate stages it has a purple or a pink colour. Both this and the proper halo show a very curious appearance of activity near their centres which strongly reminds one of the appearance in a spinthariscopes. This is no doubt due to the drops slowly falling down through the liquid.

Looking at the surrounding objects in daylight through the hot mixture, the general transparency is found to be very great, the range of spectral transmission being rather considerable. Observed with a direct vision spectroscopic, the transmitted light appears not as a sharp band as in fig. 1 in the plate but much wider. Similarly a spectrum seen through this mixture when hot shows a much broader region in good focus than does the glass powder. This is no doubt due to the smaller difference in the dispersive powers of the constituents of the mixture. The same is true of the halo also. The dark bands corresponding to figs. 2-5 are obtained but they are never so narrow.

As the small drops fall through the liquids, something like a boundary of separation forms between the clear turpentine above and the collection of drops below. This boundary like the one in the case of the glass powder shows brilliant colours different from the rest of the mixture and also different on the two sides. When, however, the drops become large, there is nothing like a regular boundary and no such effects are produced. Individual drops no doubt produce certain diffraction effects and their edges shine out with various colours depending on the difference of refractive indices of the two liquids and the angle of observation. These peculiar effects are visible not only in the drops just below the clear turpentine, but also in practically all other drops. This gives rise to a sparkling appearance in the liquid and is particularly noticeable when the mixture is just taken out of boiling water and its temperature is rather high. These effects will be discussed in a separate paper.

It may be worth while also to record the results observed when glycerine is shaken up with rather a small quantity of

turpentine. In this case, a milky emulsion of the two liquids results, which takes a very long time to clear and which may, therefore, be very conveniently used for observations of the effect of warming on its transparency. Partly on account of the very fine state of subdivision in the mixture thus obtained and partly no doubt also on account of the small quantity of turpentine taken up by the glycerine, the region in the spectrum transmitted by it when warmed up is very great in this case.

VI.—THE COLOURS OF DOUBLY REFRACTING POWDERS.

Christiansen mentions in his paper that doubly refracting powders do not give satisfactory results in his experiment, as the colours they show are not so pure, and the mixture is never fully transparent. On the other hand, we find that Prof. R. W. Wood remarks that the best results that he could obtain were with powdered quartz. It is difficult to reconcile these contradictory statements, for although it may be true that so far as the general appearance of the powder and the halo is concerned, powdered quartz does perhaps give more attractive and lively colours than does glass and the halo is also much wider, yet there does not remain the slightest doubt after a closer examination of the phenomena that Christiansen was absolutely correct in his statement referred to above. We find in fact that in the case of quartz the whole of the light that emerges through the powder appears in the halo and there is, strictly speaking, no light regularly transmitted and capable of forming a definite image of the source. The real Christiansen phenomenon is as a matter of fact altogether absent. This is apparent on observing a small brilliant source—the filament of an electric lamp or a narrow aperture backed by an arc lamp—through a layer of the mixture containing the quartz powder, when it will be found that even with very small thicknesses used, there is no defined image of the source. It remains invisible and we merely see a halo surrounding its position the colours of which gradually alter as we proceed from the centre outwards. It would therefore seem that what Prof. Wood calls the “transmitted colour” was merely the central portion of the halo and he could not have seen the actual image of the lamp flame.

The spectroscopic examination of the phenomenon in the manner described in the preceding sections makes this all the

more clear. None of the appearances shown in figs. 1-5 in the plate can be observed with quartz powder. Even in the direction of the incident light almost the complete spectrum is visible with those portions of it somewhat more intense for which the refractive index of the liquid approaches that of quartz. In directions more and more oblique to the incident light, this brighter portion gradually loses, while the portions on either side of it gain in intensity, but the difference between them becomes appreciable only at rather large angles. The colours observed either in the direction of the incident light or far out in the halo are therefore all impure by the admixture of a fair quantity of white light.

When the cell containing the mixture of powdered quartz was placed between the collimator and the prism of the spectroscope, no portion of the spectrum could be seen. There was only a general illumination of the field.

The complete absence of the transmitted light is, however, best illustrated by fig. 9 in the plate which is the ordinary spectrum as seen through the quartz powder. Fig. 8 is the spectrum seen through the glass powder under similar circumstances. It will be noticed that while in fig. 8 there is a certain region of the spectrum which appears to be well focussed and free from halo, there is no such region in fig. 9. The halo does no doubt attain a minimum size, but it does not vanish at any place. And this happens even for very small thicknesses of the powder.

This result may appear to be surprising at first sight, but is easily explained. For, on account of the doubly refracting nature of the quartz, and the absolutely random orientation of the particles with respect to their crystallographic axes, no ray can travel through the mixture from start to finish with one definite velocity. Whenever it will encounter a particle of quartz, it will in general be broken up into two, travelling with different velocities and in some cases when although it is not broken up into two, an ordinary ray may be wholly transmitted as an extraordinary ray. So that every ray must be regarded as partly ordinary and partly extraordinary during its passage through the powder. No possible value of the refractive index of the liquid mixture can allow such a ray to pass as through a homogeneous medium.

The question then arises, "To what value of μ' does the minimum size of the halo in fig. 9 correspond?" In order to settle

this, a wave-length spectroscope with a shutter eye-piece was used. The shutter was closed down to a narrow slit so that the light passing through it had a definitely known wave-length which could be quickly altered. With such an arrangement the wave-length corresponding to the minimum halo could be quite accurately determined.

A suitable refractometer not being available, the refractive index of the liquid corresponding to the minimum halo was found out by a rather indirect but quite a simple method. A tiny little prism of the same quartz (one of the broken pieces) was suspended in the liquid above the powder. Looking through this, one could see two images of the slit in the shutter eye-piece. It was easy to find which of them corresponded to the ordinary ray, for in the first place μ_0 is smaller than μ_e , and secondly the ordinary image did not move as the prism was turned, while the other moved so that the distance between them altered. By turning the wave length screw, it was easy to make the ordinary image coincide with the slit as seen directly from above and below the prism. For this value of λ , $\mu' = \mu_0$. From the known values of the refractive indices of carbon disulphide and benzol for this wave-length the proportion of the two liquids in the mixture was calculated¹ so as to satisfy the condition $\mu' = \mu_0$. The proportion having been found, the μ' for the wave-length corresponding to the minimum halo could be calculated. A specimen result is quoted below :—

minimum halo at	$\lambda_1 = 5270$
$\mu' = \mu_0$ at	$\lambda_2 = 5490$.
But	$\mu = 1.5454$ at λ_2
\therefore	$\mu' = 1.5454$ at λ_2
and	$= 1.5488$ at λ_1 .

Theoretically one would expect on an average half the path of a ray through the powder to be ordinary and half extraordinary. So that if μ' is equal to the mean of the ordinary and the mean extraordinary indices, or in other words if

$$\mu' = \frac{1}{2} \left\{ \mu_0 + \frac{1}{2} (\mu_0 + \mu_e) \right\},$$

¹ See Prestion : Theory of Light, page 134.

the path difference between a ray passing through the liquid and one through the quartz may be expected to be the least. This value of μ' should, therefore, give rise to the minimum halo. Now, for

$$\lambda = 5270, \mu_0 = 1.5465 \text{ and } \mu_e = 1.5551$$

$$\therefore \mu' = \frac{1}{2} \left\{ 1.5465 + \frac{1}{2} (1.5465 + 1.5551) \right\} = 1.54865$$

in complete agreement with the value obtained experimentally.

VII.—SUMMARY AND CONCLUSION.

The paper describes the results of a detailed study, both experimental and theoretical which has been made of the phenomena observed in Christiansen's experiment in which the powder of a transparent substance is immersed in a mixture of carbon disulphide and benzol having a refractive index nearly equal to that of the powder.

(1) *Transmitted Light.*

- (a) The observable range of wave-lengths transmitted by the mixture has been found to depend on the intensity of the incident light.
- (b) A theoretical treatment based on the principles of the wave-theory and the theory of probability has been given, and it shows that the intensity of the transmitted light is given by the expression

$$I = e^{-cd \left(\frac{\mu - \mu'}{\lambda} \right)^2}$$

The influence of the size of the particles and the thickness of the medium on the intensity of the transmitted light has been confirmed experimentally and the value of the absolute constant c has been determined and found to be roughly equal to about 7.

- (c) The formula shows that with the very finest powders, the range of transmission may be very considerable.

(2) *The Colours of the Halo.*

The statement made hitherto that the colour of the halo is complementary of that of the transmitted light is not correct. The

halo is not of one colour throughout, but has a definite structure. This is shown by observations and photographs of the spectrum of the halo, which at small obliquities consists of two narrow bright bands separated by a dark interval. These bands widen out as the angle of observation is increased. The facts are explained by observations in monochromatic light which indicate that the distribution of intensity in the halo may be taken to be of the form

$$e^{-\frac{b}{(\mu-\mu')^2} \cdot x^2}$$

which is of the exponential type, where b increases as the size of the particles is increased and as the thickness of the layer or the wave-length of light is diminished.

(3) *The Surface Colours.*

The level surface of separation between the clear liquid on the top and the powder below exhibits remarkably brilliant colours which are not only different at different stages of the mixture as regards the refractive indices, but are also different on the two sides of the boundary. This effect does not appear to have been noticed so far. Presumably it is due to a sort of total reflection from the boundary.

(4) *Liquid Mixtures.*

A mixture of glycerine and turpentine shows similar phenomena, but these become evident only when the flask containing the mixture is heated by immersion in hot water. At ordinary temperatures there is no light transmitted. The halo in this case is, however, much smaller, but the general transparency of the mixture is very great. Some interesting diffraction effects by the edges of the liquid drops are also observed.

(5) *The Colours of Doubly Refracting Powders.*

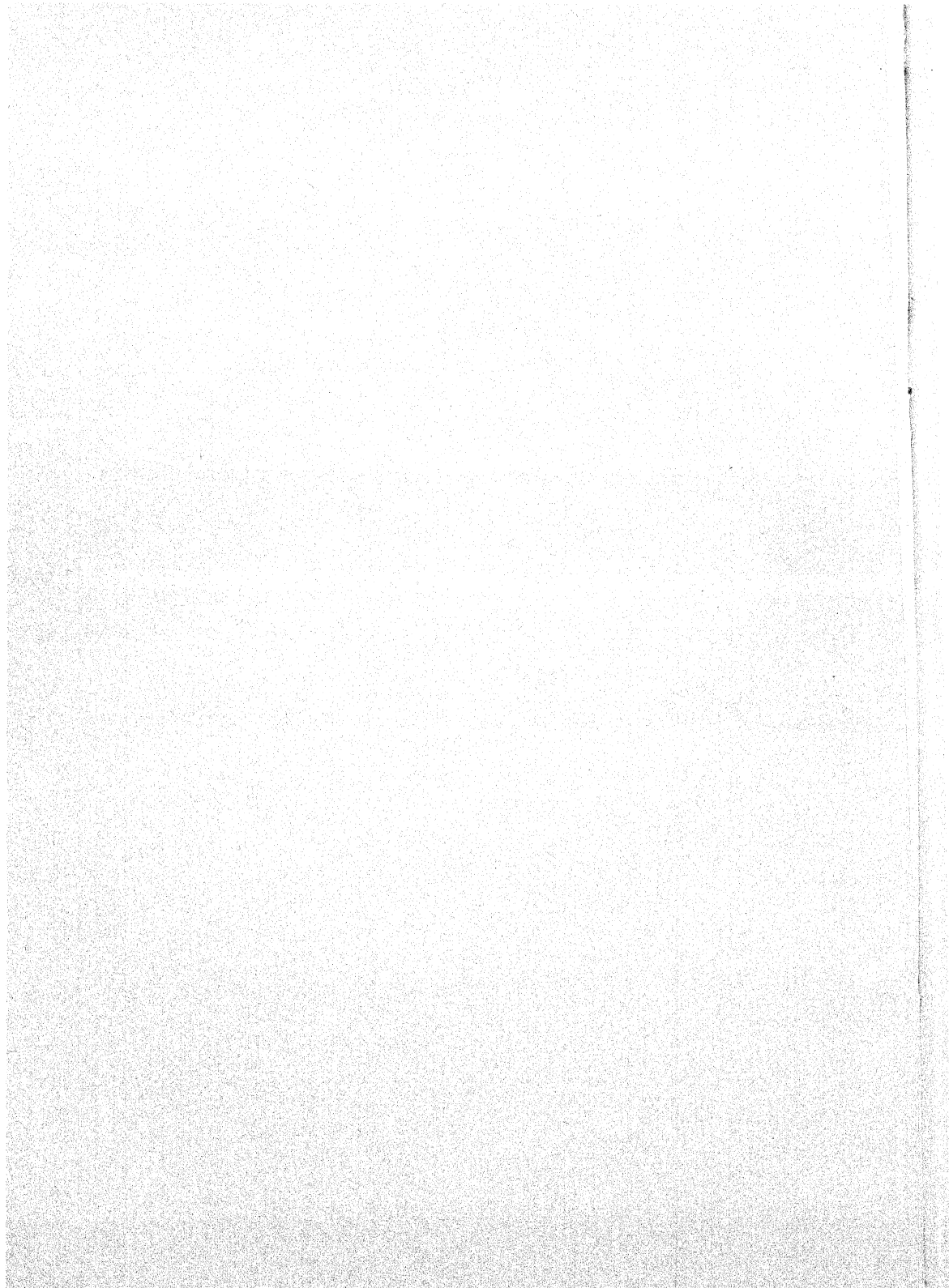
Doubly refracting powders, such as quartz, are, contrary to the statement of Prof. R. W. Wood, unsuitable for the exhibition of the true Christiansen phenomenon, because in their case there is no regularly transmitted light. The whole of the light emerging through the powder appears in the halo, which is differently coloured in its different parts. There is a particular wave-length

for which the size of the halo is a minimum and this has been shown both experimentally and theoretically to correspond to the case in which the refractive index of the liquid is equal to

$$\frac{1}{2} \left\{ \mu_0 + \frac{1}{2}(\mu_0 + \mu_e) \right\}.$$

In conclusion the writer has much pleasure in recording his indebtedness to Prof. C. V. Raman who took considerable interest throughout the progress of the work and whose suggestions and criticisms were of immense help. The experimental work was carried out at the Laboratory of the Indian Association for the Cultivation of Science.

Dated Calcutta, }
The 2nd of November 1920. }



XI. On the Production of Musical Sounds from Heated Metals.

By B. N. Chuckerbutti, M.Sc., Assistant Professor of
Physics, Calcutta University.

(Plates VIII to X.)

CONTENTS.

- I.—Introduction.
- II.—The Vibration-Curves of the Trevelyan Rocker.
- III.—Description of the Observed Phenomena.
- IV.—Crucial Test of Davis's Theory.
- V.—Summary and Conclusion.

I.—INTRODUCTION.

That vibrations giving rise to musical sounds may be produced under suitable conditions by the contact of metals at different temperatures, has long been known as a fact of observation and is best illustrated by the familiar piece of apparatus known as Trevelyan's Rocker. As is well known, this apparatus usually consists of a bar of metal (generally brass) prismatic in form which has a narrow longitudinal groove cut along its lower edge, so that if it be placed on a table, the metal rests on two adjacent parallel ridges. If displaced from its position of equilibrium, the bar rocks to and fro, resting alternately on the two lines of support, oscillating at first slowly and then more quickly as the amplitude dies away, the energy being gradually dissipated in the succession of impacts of the points of support upon the table. In actual use the rocker has a handle which consists of a brass rod with a ball at one end. The rocker is heated and is then supported in a slanting position with the edges of the groove touching the clean cold surface of a horizontal block of lead, the end of the handle resting on the

table. The rocker has generally to be started, that is, one edge has to be lifted out of contact with the lead block and then allowed to come down. The movement then continues to sustain itself, a rough sound being heard at first which later gives place to a tone of markedly musical character.

The generally accepted explanation of the working of the Trevelyan rocker (due in the first instance to Sir John Leslie) was developed by Faraday.¹ The correctness of Faraday's explanation was questioned by Forbes,² but was supported by Seebeck³ and later by Tyndall,⁴ who carried out an extensive series of studies on the subject. An experiment illustrating Faraday's theory was also made by Page⁵ in which the preliminary heating of the rocker is dispensed with and replaced by the local heating at the points of contact produced by the passage of an electric current during the experiment. Briefly stated, the view now generally held regarding the mode of action of the apparatus is as follows:—

(1) The rocker moves under the action of gravity (its own weight), resting alternately upon two ridges of support.

(2) This movement is sustained by the local periodic expansions of the lead block (which has a high rate of expansibility with increasing temperature and a low heat-conductivity), in consequence of the intermittant contact with the heated metal. The forces brought into play by these expansions do work and supply the energy requisite for the maintenance of the motion. Davis⁶ examined the subject mathematically in a paper which we shall have occasion to refer to again, and came to the conclusion that the foregoing explanation of the maintenance of the motion was adequate.

The present work was undertaken with a view to determine by direct experimental study (and not merely on *a priori* suppositions) the manner in which the heated bar vibrates and gives rise to musical tones when placed in contact with the cold metal. The

¹ Faraday: Experimental Researches in Chemistry and Physics, page 311.

² Forbes: Phil. Mag., Vol. IV, pp. 15, 182, 1834.

³ Seebeck: Pogg. Ann., Vol. LI, p. 1, 1840.

⁴ Tyndall: Sound, third edition, page 52, ed. 1875; Phil. Trans. Roy. Soc., Parts I-II, 1854.

⁵ Page: Silliman's Journal, p. 105, 1850.

⁶ Davis: Phil. Mag., Vol. XLV, p. 296, 1873.

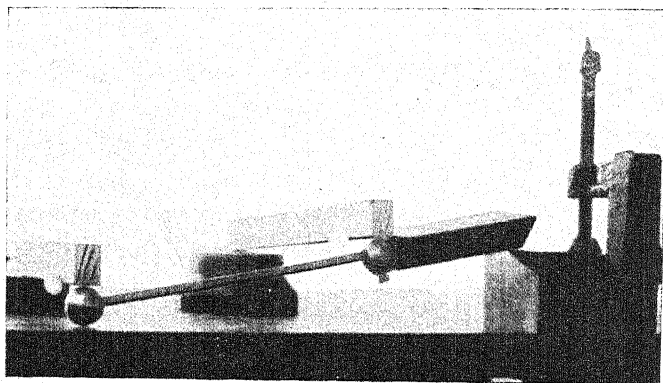


Fig. 1

Trevelyan Rocker arranged to photograph Vibration Curves.

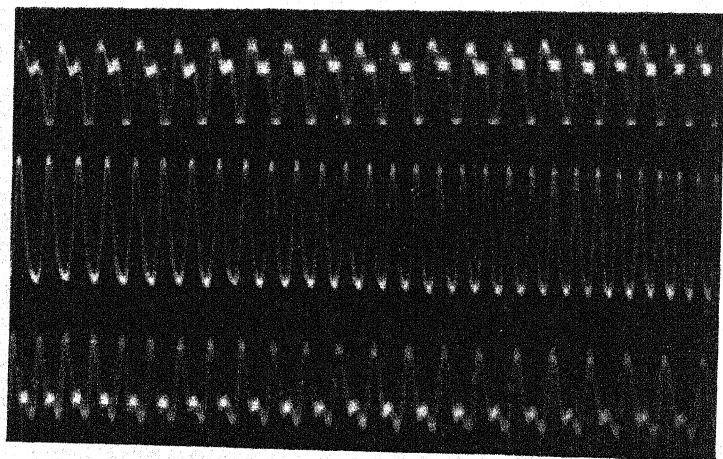


Fig. 2

Fig. 3

Fig. 4

Vibration Curves showing effect of Asymmetric Pressure.

outcome of the research has been to show that the view usually held regarding the mode of action of the apparatus requires to be modified in essential particulars.

II.—THE VIBRATION-CURVES OF THE TREVELYAN ROCKER.

The motion of the rocker in the later stages of the experiment, when it gives rise to musical tones, is of quite a small amplitude and requires to be highly magnified to be satisfactorily observed or photographed. The method adopted by the writer has proved itself to be exceedingly simple and effective in practice and is shown in fig. 1 (Plate VIII). A sewing needle is placed horizontally on the curved upper surface of a small metal block which is firmly attached to the rocker. When the rocker vibrates, the needle rolls between this surface and a small bar of wood which is placed lightly resting on the upper surface of the needle, the other end of the bar being supported at a suitable level above the surface of the table. The rotation of the needle is indicated by a mirror attached to it, on which is incident a pencil of light from a pin-hole illuminated by the electric arc. The light reflected from the mirror is focussed by a lens upon a moving photographic plate or by a suitable arrangement can be projected upon a screen to make the form of the vibration-curves visible. Magnification of the movement of the rocker ranging from 1,000 to 20,000 times may be readily obtained in this way.

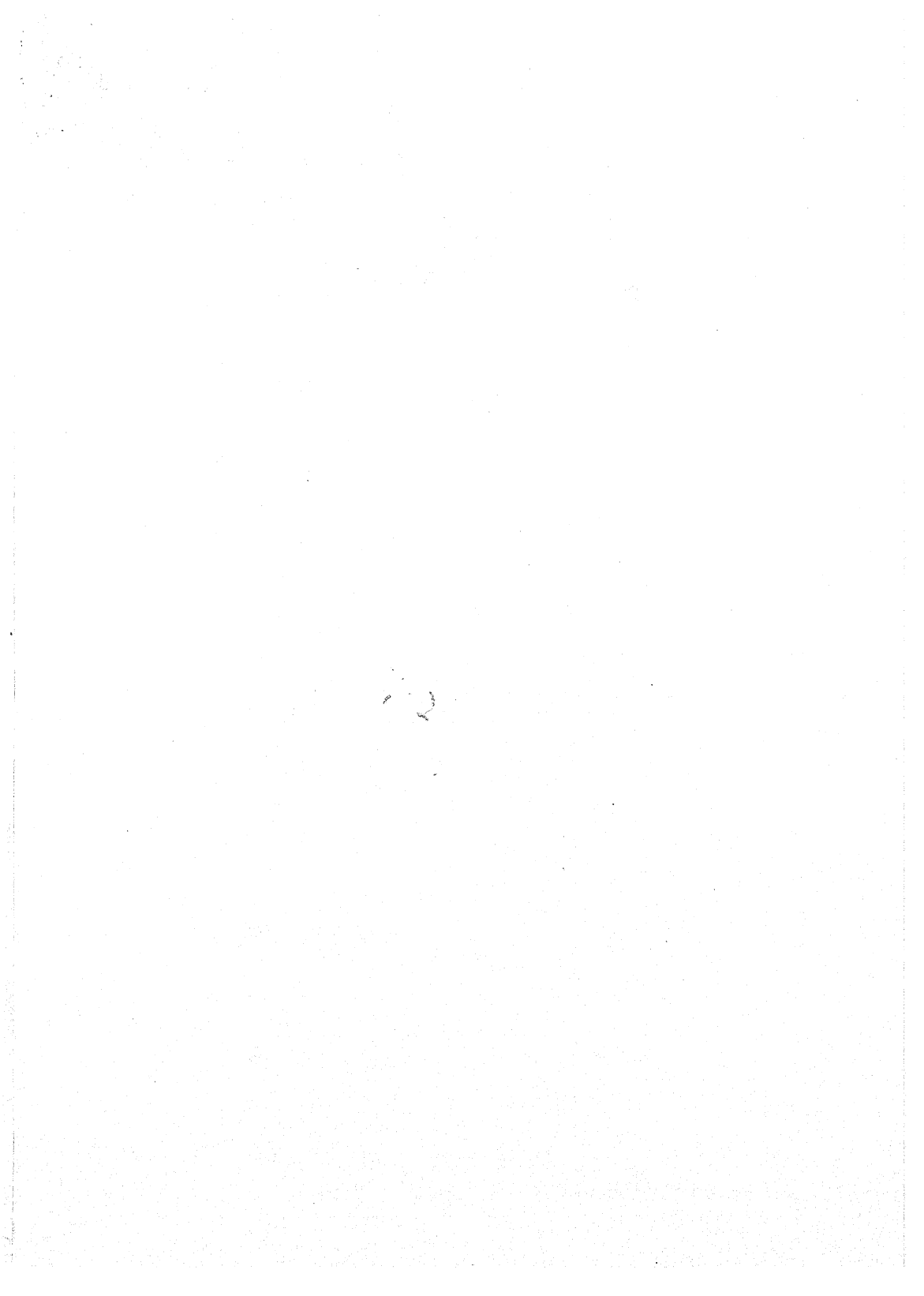
By fixing the metal block on which the needle rolls firmly to the rocker itself, the motion of the latter is readily observed. On the other hand, the block may be fixed, if desired, at any position on the handle of the rocker so as to enable the motion of the latter to be determined. The motion of the upper surface of the ball terminating the handle of the rocker may be similarly observed by a rolling needle and mirror arrangement. Simultaneous observation is also possible of the motion of the end of the handle and of the rocker itself, or of any intermediate point on the former.

III.—DESCRIPTION OF THE OBSERVED PHENOMENA.

Using the method described above, numerous observations were made and photographs secured of the mode of vibration of the heated rocker under various conditions, and several quantitative determinations of the frequency and amplitude of vibration

were also obtained. Two different rockers of the usual prismatic form, one rather massive and another of medium size, were used, and some observations were also made with a light rocker having a flat lower surface (instead of a grooved edge). Different lengths and diameters of the handle-bar were employed and the effect of loading the handle-bar at different points was also investigated. Heating the rocker to a greater or less extent, and especially the effect of exerting pressure on the surface of the rocker or on the handle were studied. Observations were also made using a block of rock-salt (which is recommended by Tyndall) instead of lead as the support for the heated rocker. It would be unprofitable to attempt to describe everything that was noticed in the course of the experiments which extended for over four months. *The main result which emerges from the work is this, that the motion of the rocker, especially in the cases in which it gives rise to musical tones, is very far indeed from being a simple rocking movement; the elastic vibrations of the rocker and handle-bar play an important part, and so far as can be ascertained, entirely replace any such simple rocking movement as may occur in the earlier stages of the experiment.*

The photographs reproduced as figs. 2 to 19 in plate will give an idea of the complexity of the vibration curves. These were all secured with the massive rocker. The noteworthy feature is that the vibrations, when steady, are generally accompanied by harmonic overtones which are of greater or less intensity according to circumstances. In the initial stages, when the temperature difference is very great, a coarse rocking movement of very large amplitude may be observed, but this is of low frequency and hardly gives rise to any audible musical tone. At a slightly later stage, the motion becomes of smaller amplitude, accompanied by partials evidently due to the elastic vibrations of the system and is often rather irregular in character. At a still later stage, the vibrations become very regular and are of markedly musical character. Figs. 5 to 9 represent the successive stages, the vibrations being those of the *end* of the handle bar which was 23 cm. long and 6.5 mm. in diameter. Figs. 10 and 11 are similar photographs with a handle bar 14 cm. long and figs. 12 and 13 with one of 9 cm. length only, the diameter of the handle bar being the same throughout. It is found that the harmonics tend to disappear in the last stages of the motion. Figs. 14, 15 and 16,



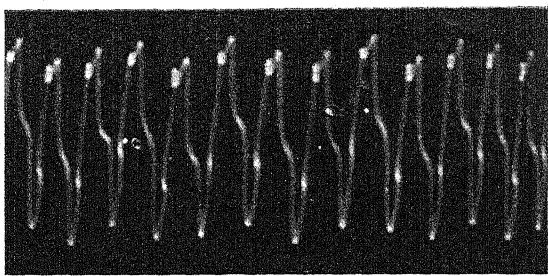


Fig. 5

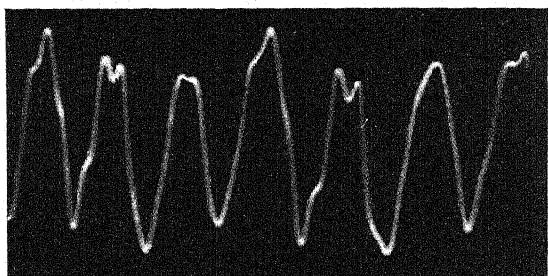


Fig. 6

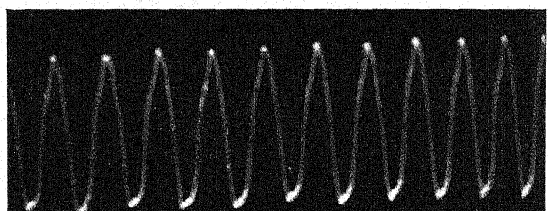


Fig. 7



Fig. 8

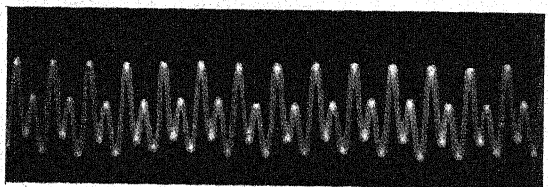


Fig. 9

The successive stages of vibration of the end of the

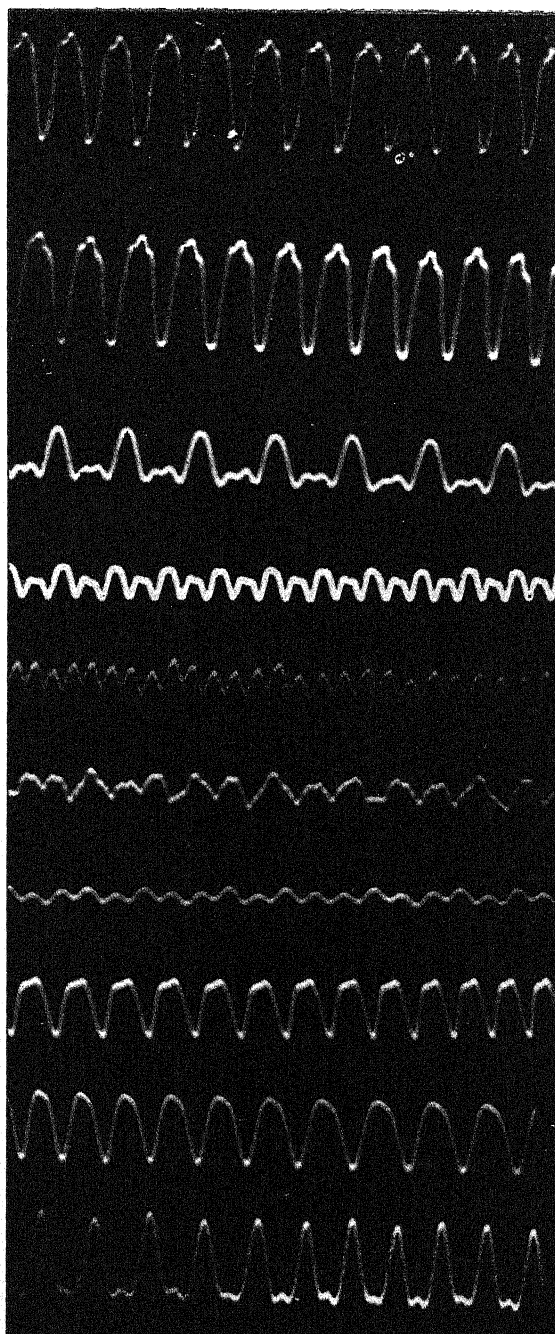


Fig. 10

Fig. 11

Fig. 12

Fig. 13

Fig. 14

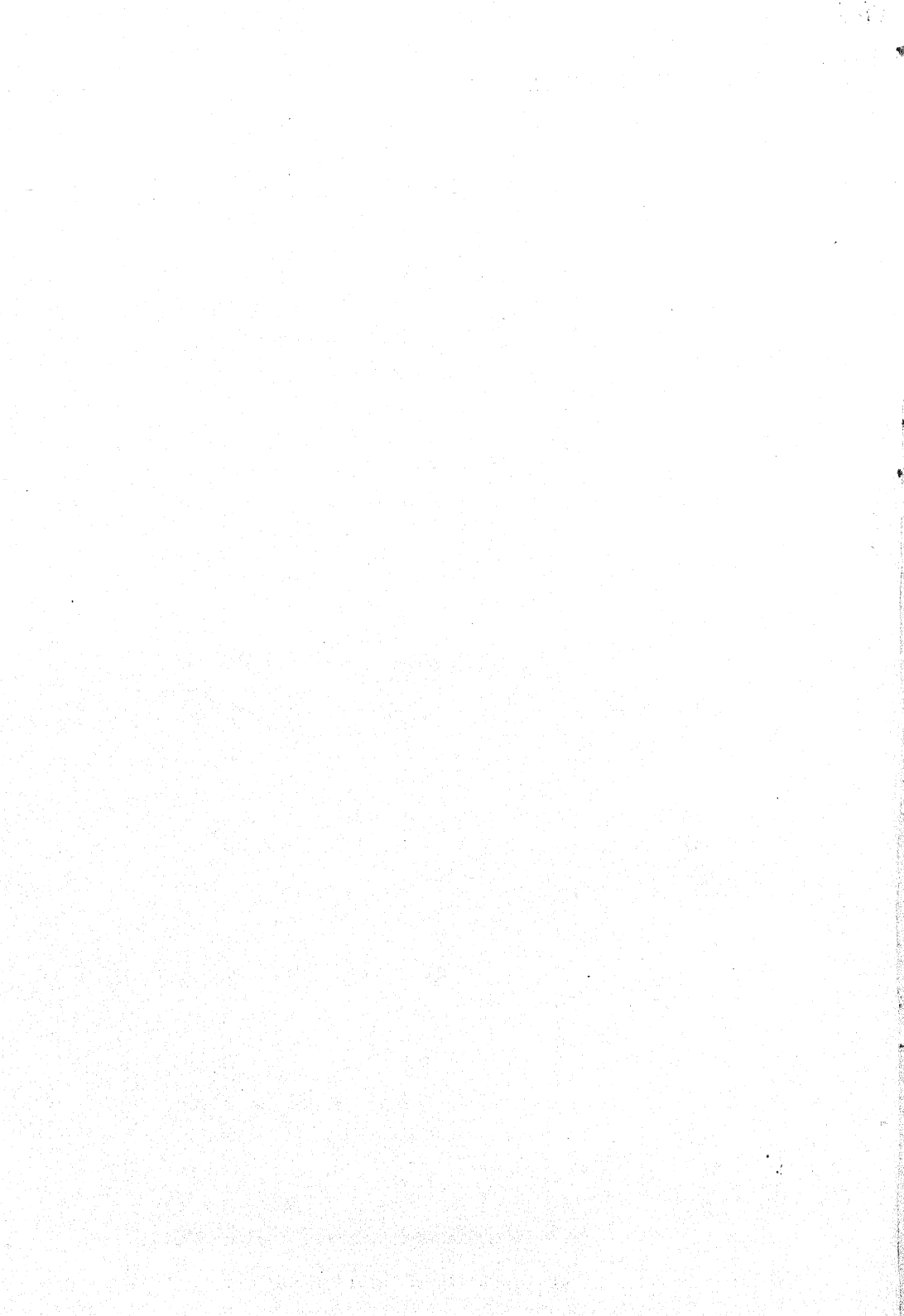
Fig. 15

Fig. 16

Fig. 17

Fig. 18

Fig. 19.



however, give the vibration curves of the block when firmly clamped to the rocker itself, the handle bar being the longest of the three. It will be seen that the motion of the rocker is, in this case, of much smaller amplitude than that of the end of the handle bar and is rather irregular in character, and is also accompanied by over-tones of high frequency which are not perceptible in the motion at the end. Figs. 2, 3, 4 illustrate an interesting effect. Fig. 3 shows the motion at the end of the handle, when the rocker was vibrating freely. Fig. 2 when the rocker was pressed down by a pointed rod so that the pressure on the lead block was greater on one of the two edges of the groove. Fig. 4 when the pressure was similarly greater on the other edge of the groove. It is seen that the phase of the harmonic is reversed in fig. 4. Figs. 17, 18 and 19 illustrate the motion at the end of the handle bar when the lead block was replaced by a block of rock salt.

The following significant facts were noticed in the course of investigation:—

1. By simultaneous observation of the motion of the rocker and of different points on the handle bar, it is found that the vibration curves at different points differ greatly in amplitude and character. The motion at the end of the handle is always of greater amplitude and indeed sometimes much greater than that of the rocker itself. In the course of vibration, the system divides itself into segments. The points of minimum motion are not however absolutely at rest but show the overtones very prominently. The relation between the movements at different points is not constant but changes when the pitch of tone emitted alters.

2. The musical tone given by the rocker when it is vibrating freely does not change pitch quite gradually but generally passes more or less abruptly from one pitch to another, the amplitude of vibration also changing suddenly at the same time.

3. The effect of pressing the rocker down on the block is always to raise the pitch; but in some cases, the amplitude of oscillation of the rocker actually increases with the rise of pitch, and in other cases decreases. When the amplitude increases with the rise of pitch as the rocker is pressed down, the loudness of the tone markedly increases.

4. Even a slight pressure on the end of the handle bar which rests on the table sometimes results in a marked rise in the pitch

of the tone of the rocker. For instance, in the case of the massive rocker with the longest handle, a pressure of only 30 gms. weight resulted in an increase of pitch from 140 to 180 vibrations per second.

5. When the rocker vibrates freely, the lower and higher limits of pitch of the musical tone which it gives rise to are determined by the mass of the rocker as well as by the length of the handle bar and its cross section. The shorter or the thicker the handle bar, the higher the pitch of the musical sound when it begins and the higher when it ends. Diminishing the mass of the rocker raises the pitch of the tone. The lower end of the range of pitch covered by the musical tones of the freely vibrating rocker is generally about the same as or a little less than the pitch of the tone given by its elastic vibrations in the gravest mode. The higher end of the range of pitch is generally rather ill-defined.

IV.—A CRUCIAL TEST OF DAVIS'S THEORY.

The fundamental assumption on which Davis (in the paper already cited) bases his investigation is that the motion of the heated rocker is a simple rocking movement about the edges under the action of its own weight. The observations and photographs of the vibration-curves of the rocker discussed above render this view untenable and tend to show indeed that it is the elastic vibrations of the system that are maintained and that determine the pitch of the tone emitted. The measurements of the amplitude and frequency of the rocker furnish a crucial test of Davis's theory which we now proceed to discuss.

The motion of the rocker alternately about its two edges under its own weight is not an isochronous oscillation. Its amplitude decreases (in the absence of a maintaining force) continually on account of the loss of energy resulting from the impacts on the edges; and the intervals between successive impacts also continually decrease. The relation between the period and the amplitude of the movement of the rocker at any stage may be determined from the equation of motion which to a first approximation is

$$MK^2 \frac{d^2\theta}{dt^2} = -Mga$$

where M = mass of the rocker,

K =radius of gyration about either of the edges of support,
and $2a$ =Distance between the edges.

Hence, integrating,

$$K^2 \int_{t=0}^{t=T/4} \frac{d^2\theta}{dt^2} dt = -ag \int_{t=0}^{t=T/4} dt$$

remembering that, when

$$t=0, \frac{d\theta}{dt} = \omega$$

$$K^2\omega = ag T/4 \dots\dots\dots (1)$$

Now, θ being the greatest angle which the plane containing the edges makes with the horizontal surface of the block

$$\theta = \frac{R}{2a}, \quad R \text{ being the amplitude.}$$

Hence, since ω increases from 0 value to ω in the interval $T/4$

$$\frac{\omega}{2} = \frac{2R}{aT} \quad \text{or} \quad \omega = \frac{4R}{aT}.$$

Substituting for ω in (1)

$$T^2 = \frac{16RK^2}{a^2g} \dots\dots\dots (2)$$

On the left hand side of this equation, the period of oscillation T of the rocker may be determined experimentally by finding the pitch of the tone given by it with the sonometer or by recording its vibration curves simultaneously with those of a tuning fork. On the righthand side, all the quantities are known except the value of R which may be determined from the observation of the magnified amplitude of the vibration of the rocker. For if I be the movement of the spot of light which we observe and d be the distance of the focus from the lens, r the radius of the rolling needle, then the displacement of the upper surface of the curved mass of metal upon which the needle rolls is given by $\frac{rI}{2d}$.

Hence, knowing the distance of this surface from the axis of rotation of the rocker and the distance between the two ridges, we may deduce the magnitude of the vertical displacement of the free edge from the surface of the block. Thus it is possible to determine directly the quantities on either side of equation (2) and to test its

validity and therefore also the validity of Davis's theory. In the table below, the first column gives the serial number of observation (the handle of the rocker being of different length in some of these cases). The second column gives the frequency of the tones as determined by the sonometer. The third gives the value of the magnified motion (1), the fourth column gives the displacement of the upper surface of the curved mass of metal, the fifth column gives the vertical movement R of the edge of the rocker, and in the sixth column we have the value of the frequency calculated from the equation (2). It will be noticed that the observed and calculated frequencies differ very widely, and it is clear, therefore, that the frequency of the oscillation is not determined by the motion of the rocker under the action of its weight. To exhibit this even more clearly, column 7 of the table shows the value for I , the movement of the spot of light, that would be necessary to give the observed frequency of oscillation if equation (2) were to hold good. It will be found on comparison of column (3) and (7) that the observed oscillation is of far smaller magnitude than deduced from the observed frequency.

DIMENSIONS OF THE ROCKER.

$2a$ = Distance between the ridges

= 0.6 cm.

Base of the triangular section

= 5 cm.

Length of each side of the triangle

= 3.4 cm.

Height of the C.G. above a line joining the ridges

= b = 1.41 cm.

K^2 = 3.47

(Radius of the needle r = 0.042 cm.

d = 98 cm.)

1	2	3	4	5	6	7
No. of Observations.	Frequency by sonometer.	I (in cm.).	Displacement $= \frac{rI}{2d}$ cm.	Vert. displacement of the ridge. (in cm.).	Frequency (calculated).	Value for I that would fit in equation (2).
1	115	0.75	0.00016	0.000018	301	5.05 cm.
2	121	0.70	0.00015	0.000017	311	4.7 cm.
3	130	0.64	0.00013	0.000015	331	4.0 cm.
4	136	0.60	0.00012	0.000013	342	3.62 cm.
5	145	0.50	0.00010	0.000011	381	3.20 cm.
6	162	0.40	0.00008	0.000009	420	2.6 cm.
7	180	0.20	0.00004	0.000005	595	2 cm.

The foregoing conclusions are not invalidated on taking into account the maintenance of the motion by the periodic expansions and contractions of the block of lead on which the rocker rests. For, to a first approximation, the period of the maintained oscillation must be the same as that of the unmaintained motion having the same amplitude. It can readily be shown that according to Davis's theory, the actual expansions and contractions of the lead block necessary to maintain the oscillation are small compared with the motion of the free edges of the rocker. This may be done in the following way:—

Let ω and ω' be the angular velocities just before and just after impact of the rocker upon one point of the lead block (say O). Then the relation connecting them is¹

$$\omega = \frac{K_1^2 + b^2 + a^2}{K_1^2 + b^2 - a^2} \times \omega',$$

K_1 being the radius of gyration of the rocker about the centre of gravity. If the vibrations are continuous, the angular velocity acquired just before the next impact upon the other point of support (say O') must be equal to $-\omega$. Immediately after the impact upon O , the point begins to rise. Suppose, it has risen through a

¹ Davis, loc. cit.

height h in time t . Then it can be shown that the equation of motion of the rocker is

$$\left(K_1^2 + a^2 + b^2\right) \frac{d^2\theta}{dt^2} = -a \left(g + \frac{d^2h}{dt^2}\right)$$

that is

$$K^2 \frac{d^2\theta}{dt^2} = -a \left(g + \frac{d^2h}{dt^2}\right)$$

where K_1 = radius of gyration about an axis passing through the centre of gravity,

and K = radius of gyration about one of the edges of the rocker.

Hence integrating and noticing that

$$\text{when } t=0, \frac{d\theta}{dt} = \omega'$$

we have

$$K^2 \frac{d\theta}{dt} = -agt - a \frac{dh}{dt} + K^2 \omega'$$

Integrating again and noticing that

$$\text{when } t=0, h=0, \text{ and } \theta = \frac{H}{2a}$$

(H being the height through which O rises),

$$\text{we have } K^2 \theta = -\frac{agt^2}{2} - ah + K^2 \omega' t + K^2 \frac{H}{2a}.$$

Now, just before the next impact O will have risen to a height H , and O' will have sunk down to its original position. Therefore

the value of θ and $\frac{d\theta}{dt}$ will be $-\frac{H}{2a}$ and $-\omega$ respectively. The time

for which the rocker is in contact with the block being $T/2$ where T = period of vibration, the above relation takes the form,

$$K^2 \frac{H}{a} = \frac{agT^2}{8} + aH - K^2 \omega' \cdot \frac{T}{2}$$

whence

$$\omega' = \frac{a^2 g T^2 + 8H(a^2 - K^2)}{4aK^2 T}$$

or

$$e.\omega. = \frac{a^2 g T^2 + 8H(a^2 - h^2)}{4aK^2 T}$$

where,

$$c = \frac{K_1^2 - a^2 + b^2}{K_1^2 + a^2 + b^2}.$$

Substituting in this equation, the value for ω as obtained before, viz.

$$\omega = \sqrt{gT/4} K^2$$

$$H = \frac{a^2 g T^2 (1-c)}{8(2-aK^2)} \dots \dots \dots (3)$$

From equations (2) & (3) we have

$$\frac{R}{H} = \frac{K^2 - a^2}{2K^2(1-c)} = \frac{3.47 - .09}{3.47 \times .1} = 9.7$$

The free motion of the edges of the rocker required on Davis's theory is thus about ten times the expansion of the supporting block necessary to maintain it, and the influence of the latter on the period of the motion under gravity is therefore quite negligible. Thus we finally arrive at the conclusion that the observed amplitudes and frequencies are not connected together in the manner necessary for the validity of Davis's theory, and the discussion supports the view that the frequency of oscillation of the rocker is determined principally by its elastic vibrations.

The two facts now remaining to be explained are (1) the rise of the pitch of the tone of the rocker as the motion dies away and (2) also, the rise of pitch resulting from exerting pressure on the rocker. It does not seem difficult to reconcile these with the view that the pitch is determined by the elastic vibrations of the system. For, the contact of the rocker with the lead block at one end and of the handle with the table at the other end no doubt operates as a constraint tending to raise the pitch of the elastic vibrations to a greater or less degree depending on the effectiveness of the constraint. Exerting a pressure on the rocker by the point of a rod resting upon it would similarly operate as a constraint raising the pitch of the vibration to quite a considerable extent.

V.—SUMMARY AND CONCLUSION.

The paper describes an experimental study and a theoretical discussion of the vibrations of the Trevelyan Rocker, and the manner in which the musical tones given by the rocker arise. A method is described by which the vibration curves at different parts of the rocker may be observed and photographed and numerous illustrations are reproduced. The theory of the rocker worked out mathematically by Davis is critically discussed, and

it is shown that the basis on which it rests, namely, that the motion of the heated bar is a simple rocking about the two edges under the action of gravity is contradicted by observations on the amplitude and frequency of the vibration of the rocker. The outcome of the research is to show that the musical tones principally arise from the elastic vibrations of the system composed of the rocker and its handle, and its pitch is determined by these vibrations. As the vibrations occur under constraint, the pitch may vary to some extent with the experimental conditions. The effect of pressure on the rocker in raising the pitch is similarly explained.

In conclusion the author wishes to record his best thanks to Prof. C. V. Raman who suggested the investigation and kindly provided all facilities for work at the University College of Science.

Dated Calcutta,
The 18th October, 1920. }

XII. Some New Illustrations of Optical Theory by Ripple Motion.

By Rajendra Nath Ghosh, M.Sc.

(Plate XI.)

CONTENTS.

- I.—Introduction.
- II.—Description of the New Ripple Apparatus.
- III.—Interferences near Caustics and the Theory of the Rainbow.
- IV.—Anamolous Propagation near Foci.
- V.—Laminar Diffraction.
- VI.—Summary and Conclusion.

I.—INTRODUCTION.

Since Vincent published his well-known photographs of ripples presenting analogies with various optical phenomena, the subject has continued to attract attention from other workers who have used the same or different methods. A fairly comprehensive bibliography of the literature will be found in a recent paper by Watson and Shewhart.¹ In the present paper will be described some further studies in the same field carried out by the author while working at the Indian Association for the Cultivation of Science, Calcutta. The research was undertaken with the aid of a new apparatus which is very compact and serves as a convenient method of producing ripples and observing them by intermittent illumination, the frequency and amplitude of the ripples being both capable of being rapidly altered without interference with the stroboscopic arrangement. The design of the apparatus is due to

¹ Physical Review, February 1916, page 226.

Prof. C. V. Raman and is somewhat similar to that described by him in a recent paper on the experimental study of vibrations.¹

II.—DESCRIPTION OF THE NEW RIPPLE APPARATUS.

Fig. 1 in Plate XI is a picture of the apparatus used for exciting ripples, part of the ripple tank also appearing in the photograph reproduced. An electric motor causes a horizontal pivoted rod to oscillate rapidly in a vertical plane, a dipper hanging from the end of this rod exciting the ripples on the surface of the liquid contained in the tank seen to the right in the picture. The motor also carries a card-board disc with a sector cut out in it through which the full beam of an electric lantern can pass once in each revolution of the motor. The beam of light is reflected by means of mirrors and comes up through the glass bottom of the tank, the liquid used in it being generally water. The ripples are then seen on a screen held vertically above the tank. They may, if desired, also be projected horizontally on an ordinary lantern screen by using a mirror held at 45° above the surface of the tank to reflect the light at right angles to its path. Since the flashes of light are synchronous with the oscillations of the dipper, the ripples appear perfectly stationary. The frequency and wave-length of the ripples may be altered without interfering with the running of the apparatus by merely increasing or decreasing the speed of the motor by adjustment of a rheostat kept in circuit with it. The amplitude of the ripples excited may be adjusted to any desired value and kept constant by moving the crank-pin on the shaft of the motor to the position necessary and then fixing it. As the arrangement is very robust, any desired form of dipper may be put on without prejudice to the working of the apparatus. Since the full beam of the lantern can pass through the aperture in the stroboscopic disc, there is an abundance of light for all purposes. The device for altering the wave-length of the ripples is very useful in illustrating the theory of interference and diffraction phenomena on the screen by lantern projection. The ripples are seen perfectly at rest and can be photographed from the screen with any ordinary camera.

¹ Physical Review, Nov. 1919. See also these Proceedings, Vol. VI, Plate V.

Fig. 1

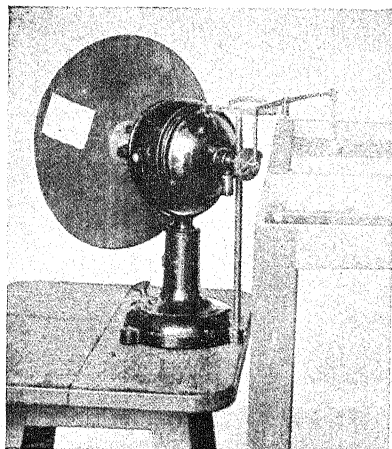


Fig. 3

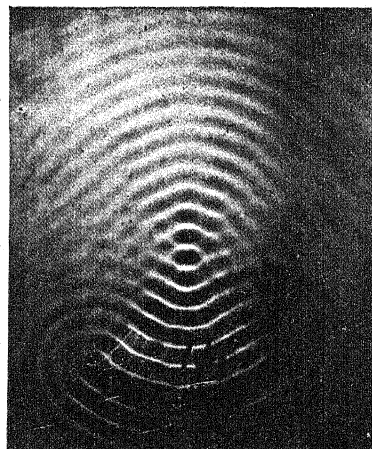


Fig. 2



Fig. 4



Ripple Apparatus and Photographs.

1999, 2000, 2001, 2002, 2003, 2004, 2005, 2006, 2007, 2008, 2009, 2010, 2011, 2012, 2013, 2014, 2015, 2016, 2017, 2018, 2019, 2020, 2021, 2022, 2023, 2024, 2025, 2026, 2027, 2028, 2029, 2030, 2031, 2032, 2033, 2034, 2035, 2036, 2037, 2038, 2039, 2040, 2041, 2042, 2043, 2044, 2045, 2046, 2047, 2048, 2049, 2050, 2051, 2052, 2053, 2054, 2055, 2056, 2057, 2058, 2059, 2060, 2061, 2062, 2063, 2064, 2065, 2066, 2067, 2068, 2069, 2070, 2071, 2072, 2073, 2074, 2075, 2076, 2077, 2078, 2079, 2080, 2081, 2082, 2083, 2084, 2085, 2086, 2087, 2088, 2089, 2090, 2091, 2092, 2093, 2094, 2095, 2096, 2097, 2098, 2099, 2100, 2101, 2102, 2103, 2104, 2105, 2106, 2107, 2108, 2109, 2110, 2111, 2112, 2113, 2114, 2115, 2116, 2117, 2118, 2119, 2120, 2121, 2122, 2123, 2124, 2125, 2126, 2127, 2128, 2129, 2130, 2131, 2132, 2133, 2134, 2135, 2136, 2137, 2138, 2139, 2140, 2141, 2142, 2143, 2144, 2145, 2146, 2147, 2148, 2149, 2150, 2151, 2152, 2153, 2154, 2155, 2156, 2157, 2158, 2159, 2160, 2161, 2162, 2163, 2164, 2165, 2166, 2167, 2168, 2169, 2170, 2171, 2172, 2173, 2174, 2175, 2176, 2177, 2178, 2179, 2180, 2181, 2182, 2183, 2184, 2185, 2186, 2187, 2188, 2189, 2190, 2191, 2192, 2193, 2194, 2195, 2196, 2197, 2198, 2199, 2200, 2201, 2202, 2203, 2204, 2205, 2206, 2207, 2208, 2209, 2210, 2211, 2212, 2213, 2214, 2215, 2216, 2217, 2218, 2219, 2220, 2221, 2222, 2223, 2224, 2225, 2226, 2227, 2228, 2229, 2230, 2231, 2232, 2233, 2234, 2235, 2236, 2237, 2238, 2239, 2240, 2241, 2242, 2243, 2244, 2245, 2246, 2247, 2248, 2249, 2250, 2251, 2252, 2253, 2254, 2255, 2256, 2257, 2258, 2259, 2260, 2261, 2262, 2263, 2264, 2265, 2266, 2267, 2268, 2269, 2270, 2271, 2272, 2273, 2274, 2275, 2276, 2277, 2278, 2279, 2280, 2281, 2282, 2283, 2284, 2285, 2286, 2287, 2288, 2289, 2290, 2291, 2292, 2293, 2294, 2295, 2296, 2297, 2298, 2299, 2300, 2301, 2302, 2303, 2304, 2305, 2306, 2307, 2308, 2309, 2310, 2311, 2312, 2313, 2314, 2315, 2316, 2317, 2318, 2319, 2320, 2321, 2322, 2323, 2324, 2325, 2326, 2327, 2328, 2329, 2330, 2331, 2332, 2333, 2334, 2335, 2336, 2337, 2338, 2339, 2340, 2341, 2342, 2343, 2344, 2345, 2346, 2347, 2348, 2349, 2350, 2351, 2352, 2353, 2354, 2355, 2356, 2357, 2358, 2359, 2360, 2361, 2362, 2363, 2364, 2365, 2366, 2367, 2368, 2369, 2370, 2371, 2372, 2373, 2374, 2375, 2376, 2377, 2378, 2379, 2380, 2381, 2382, 2383, 2384, 2385, 2386, 2387, 2388, 2389, 2390, 2391, 2392, 2393, 2394, 2395, 2396, 2397, 2398, 2399, 2400, 2401, 2402, 2403, 2404, 2405, 2406, 2407, 2408, 2409, 2410, 2411, 2412, 2413, 2414, 2415, 2416, 2417, 2418, 2419, 2420, 2421, 2422, 2423, 2424, 2425, 2426, 2427, 2428, 2429, 2430, 2431, 2432, 2433, 2434, 2435, 2436, 2437, 2438, 2439, 2440, 2441, 2442, 2443, 2444, 2445, 2446, 2447, 2448, 2449, 2450, 2451, 2452, 2453, 2454, 2455, 2456, 2457, 2458, 2459, 2460, 2461, 2462, 2463, 2464, 2465, 2466, 2467, 2468, 2469, 2470, 2471, 2472, 2473, 2474, 2475, 2476, 2477, 2478, 2479, 2480, 2481, 2482, 2483, 2484, 2485, 2486, 2487, 2488, 2489, 2490, 2491, 2492, 2493, 2494, 2495, 2496, 2497, 2498, 2499, 2500, 2501, 2502, 2503, 2504, 2505, 2506, 2507, 2508, 2509, 2510, 2511, 2512, 2513, 2514, 2515, 2516, 2517, 2518, 2519, 2520, 2521, 2522, 2523, 2524, 2525, 2526, 2527, 2528, 2529, 2530, 2531, 2532, 2533, 2534, 2535, 2536, 2537, 2538, 2539, 2540, 2541, 2542, 2543, 2544, 2545, 2546, 2547, 2548, 2549, 2550, 2551, 2552, 2553, 2554, 2555, 2556, 2557, 2558, 2559, 2560, 2561, 2562, 2563, 2564, 2565, 2566, 2567, 2568, 2569, 2570, 2571, 2572, 2573, 2574, 2575, 2576, 2577, 2578, 2579, 2580, 2581, 2582, 2583, 2584, 2585, 2586, 2587, 2588, 2589, 2590, 2591, 2592, 2593, 2594, 2595, 2596, 2597, 2598, 2599, 2600, 2601, 2602, 2603, 2604, 2605, 2606, 2607, 2608, 2609, 2610, 2611, 2612, 2613, 2614, 2615, 2616, 2617, 2618, 2619, 2620, 2621, 2622, 2623, 2624, 2625, 2626, 2627, 2628, 2629, 2630, 2631, 2632, 2633, 2634, 2635, 2636, 2637, 2638, 2639, 2640, 2641, 2642, 2643, 2644, 2645, 2646, 2647, 2648, 2649, 2650, 2651, 2652, 2653, 2654, 2655, 2656, 2657, 2658, 2659, 2660, 2661, 2662, 2663, 2664, 2665, 2666, 2667, 2668, 2669, 2670, 2671, 2672, 2673, 2674, 2675, 2676, 2677, 2678, 2679, 2680, 26

III.—INTERFERENCES NEAR CAUSTICS AND THE THEORY OF THE RAINBOW.

The geometrical theory of the rainbow given by Descartes accounted for the principal bow qualitatively. The more complete theory (involving interferences) was given by Young who thus explained the supernumerary bows. Airy showed that the wave-front emergent from the drop has the form of an inflected curve, and according to his calculations, the direction of the principal maximum of intensity is slightly different from the normal to the inflectional tangent, which is the direction (accordingly to the geometrical theory) in which the least deviated ray travels. The supernumerary bows are the interference fringes formed alongside the caustic, and they decrease gradually in width and intensity as we move away from the principal fringe.

The theory of the rainbow may be beautifully illustrated by means of ripples. For this purpose, the dipper used to excite the ripples is given the form of the inflected wave-front emergent from the drop, of course on a much larger scale. The dipper should be carefully shaped by drawing the desired form of curve and bending a strip of brass or copper to its shape, and it should just touch the surface of the liquid. The ripple as it starts out from the dipper has the form of an inflected curve, but in proceeding outwards it doubles up and forms two superposed ripple-trains which interfere, giving maxima and minima of gradually decreasing width and intensity. This effect is clearly shown in fig. 2 in Plate XI. The photograph clearly shows how the ripples from the concave half of the strip converge to a train of foci, then diverge again, and ultimately become superimposed on the ripples diverging from the convex half, giving interferences. The great width and clearness of the principal fringe, and the weaker and narrower fringes lying on one side of it are clearly shown. In agreement with theory, the minima are practically lines of zero amplitude, and the direction of the maximum disturbance is also deviated a little away from the normal to the curve at the point of inflection and towards the first minimum lying on one side of it. An interesting feature will also be noticed in the photographs that the ripples in the two regions of maximum lying on either side of each minimum are displaced relatively to each other. This occurs to a

greater or less extent in all cases of interference, and is closely connected with the question of energy flow in the field.

The intensity at any point in the interference field due to an inflected wave-front can be readily calculated by elementary methods without carrying out the usual elaborate integration over the wave-front first worked out by Airy.

Let Oy be the normal to the inflectional tangent xOx' , the equation of the wave-front being $y = Ax^3$. Let P and P' be corresponding points at which the wave-normals are parallel to each other and inclined at an angle θ to the y -axis. We may assume without appreciable error that the effects of different parts of the

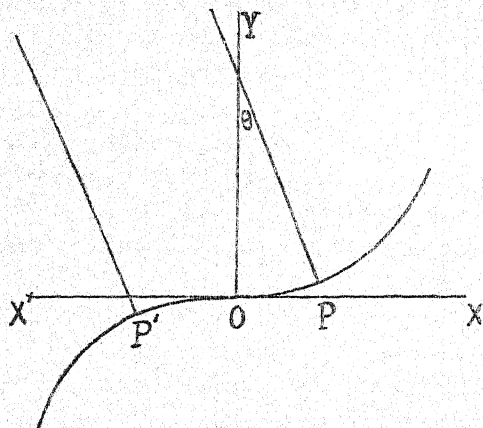


FIG. 5.

wave-front are propagated as rays in the direction of the different wave-normals in accordance with the laws of geometrical optics. The curvatures of the wave-front at P and P' are equal and of opposite sign, and hence at a sufficiently great distance the effects contributed in the direction of the wave-normals at these points will be of equal amplitude. These effects are proportional to a_1 where

$$a_1^2 = 1 / \frac{d^3 y}{dx^3} = 1 / 6Ax = \pm \left(\frac{3A}{\theta} \right)^{\frac{1}{2}} / 6A. \dots\dots\dots (1)$$

On taking the square root, the value a_1 corresponding to the negative sign becomes imaginary. We may interpret this as a gain of phase of $\pi/2$ which occurs in the rays from the part of the wave-

front which changes the sign of its curvature in passing through a focus. The numerical value of a_1 is

$$(12A\theta)^{-1/4}.$$

The path difference of the rays proceeding from P and P' can then be easily shown to be

$$\frac{2\pi}{\lambda} \left[\frac{4}{3} \theta \left(\frac{\theta}{3A} \right)^{\frac{1}{2}} \right] - \frac{\pi}{2}.$$

The amplitude of the resultant of the two will be

$$a_2 = 2(12A\theta)^{-1/4} \cos \left\{ \frac{\pi}{\lambda} \cdot \frac{4}{3} \theta \left(\frac{\theta}{3A} \right)^{\frac{1}{2}} - \frac{\pi}{4} \right\} \dots \dots \dots (2)$$

On squaring this, we get the intensity in the direction θ , and it can easily be shown that equation (2) gives results which, for positive

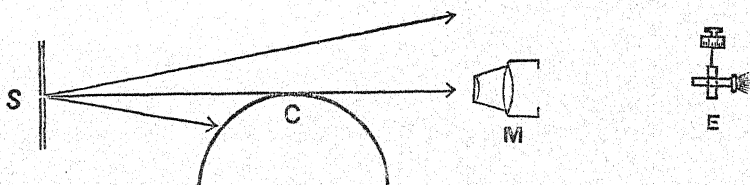


FIG. 6.

values of θ greater than a certain very small limit, are for all practical purposes the same as those given by Airy's integral. In fact the result given by (2) is identical with that obtained from Airy's integral by the method of semi-convergent expansions.

The most convenient experimental arrangement for observing on the optical bench the different steps in the transformation of an inflected wave-front to a cusped wave showing the interferences alongside the caustic is that indicated diagrammatically in fig. 6.

S is the slit which is the source from which light diverges and falls upon the reflecting cylinder C . M is a short focus lens which is placed so that its focal plane is well in advance of the edge of the cylinder C grazed by the incident rays. The rays after reflection at the surface of the cylinder form a virtual caustic within the cylinder. After passing through the lens they form an inflected wave-front which in the region behind, forms a cusped wave-front

and shows the interferences alongside the caustic very well.¹ By interposing a wire of greater or less diameter immediately behind the lens, different portions of the inflected wave-front can be cut off. As the wire is moved across the field from one side to the other, its shadow moves in the interference-pattern towards the caustic and then *retraces its path*, the fringes practically disappearing from the region covered by the shadow,² showing thereby clearly that the inflected wave doubles up and forms a cusped wave-front in the course of its propagation and thus gives rise to interferences.

The formula (2) given above for the disturbance in the interference-field due to an inflected wave-front was tested experimentally *with the ripple apparatus* by measuring the angular positions of the minima of disturbance relatively to the inflectional tangent, and also calculating the same from the observed wave-length of the ripples and the known form of the dipper. The results are shown in the table given below:—

TABLE I.

 $\lambda = 1.07 \text{ cm.}$ $A = 0.45.$

Minimum No.	1	2	3	4	5
Pontoon observed ..	66°	50°	33°	29°	21°
„ calculated ..	68°	52°	40°	28°	23°

The agreement is fairly satisfactory.

IV.—ANAMOLOUS PROPAGATION NEAR FOCI.

Gouy discovered in 1890³ that a spherical wave gains in phase by half a wave-length in passing through a focus. His experiments were repeated by Fabry⁴ and Zeeman. Later Sagnac⁵ and Reiche again performed the experiment. Sagnac on the basis of diffrac-

¹ See N. Basu, Phil. Mag., Jan. 1918, page 92.

² The fringes elsewhere are also disturbed by the diffraction effects due to the wire, but not to such an extent as to obscure the general nature of the case. Some interesting effects are observed when the wire just coincides with the point of inflection in the wave-front.

³ Gouy, Ann. de Chim et de Phys., 6th series 24, pp. 145-213.

⁴ Fabry, Journal de Phys., 3rd series 2, p. 22 (1892).


⁵ Sagnac, Journal de Physique, Oct. 1903.

tion theory explained the gain of phase as due to the oscillation of phase and intensity; Reiche¹ on the electromagnetic theory showed that the electric vector in passing through the focus changes sign while the magnetic vector remains unchanged, indicating a gain of phase of half a wave length in the passage through the focus. Since it is now established that a wave gains in phase in passing through a focus, it appeared worth while to see whether it is possible to detect the phenomenon in the case of ripples as well. With this object, the dipper was made into the shape of an arc of a circle of radius 3.5 cm. When a plane wave passes through a lens, it is transformed into a concave wave which first converges to a focus, and then diverges into the surrounding medium in the form of convex waves. The waves generated by the dipper behave similarly. Fig. 3 in the plate shows the effect. Careful observation shows that the wave lengths near the focus are slightly longer than the others. This can be interpreted as equivalent to an acceleration of velocity near the focus, so that the waves gain a certain fraction of wave-length in phase in passing through it. Using very small amplitudes of excitation of the ripples so as to avoid errors due to the alteration of the velocity of the ripples which occur at larger amplitudes, a large number of measurements were made with the ripple apparatus, the total space occupied by three waves at the focus, and three waves on either side of it being determined. The former was found to be greater than the mean of the two latter by about quarter of a wave-length, which is what we should expect in the case of cylindrical waves in the optical case. It may be remarked that this gain in phase in passing through a focus is also indicated by the good agreement between theory and experiment in the case of the rainbow (dealt with in the preceding section) where the phase-change was taken into account in finding the positions of the minima of illumination.

It may be remarked in passing that fig. 3 also clearly shows the diffraction-effects in the neighbourhood of the focus. It is precisely because of diffraction that it is necessary to take into account a sufficient number of waves on either side of the focus in order to determine the gain in phase.

¹ Reiche, *Ann. d. Physik*, April and June 1909. (An account is given in Wood's *Optics*, page 263.)

V.—LAMINAR DIFFRACTION.

In some recent studies of laminar diffraction phenomena,¹ it has been found that the edges of thin laminae often diffract light in a strongly unsymmetrical manner, a great deal more light being scattered towards the retarded side of the wave-front than on the other. It was thought that it would be of interest to try and illustrate this phenomenon with the aid of the ripple apparatus. A dipper was used of the form of a long  so as to imitate the discontinuous form of wave which presumably arises when a plane wave passes through a semi-infinite transparent lamina with a sharp straight edge. The result obtained is illustrated in fig. 4 of the Plate. It will be seen that the disturbance arising from the region of bend in the dipper is overwhelmingly large to the region on the right which corresponds to the retarded side of the wave-front, while on the left of the normal drawn to the line of the dipper, the ripples are almost perfectly straight with hardly any sensible disturbance. Reversing the position of the dipper so that the retarded side of the ripple was on the left immediately resulted in a corresponding reversal of the pattern showing that the effect was genuine.

VI.—SUMMARY AND CONCLUSION.²

The paper describes the construction and use of a very simple and compact apparatus which enables ripples to be excited on the surface of a liquid and observed stroboscopically, the frequency of the ripples being capable of rapid alteration while the apparatus

¹ P. N. Ghosh, Proc. Roy. Soc. London, A series, Vol. 96, 1919, page 261; Proc. Indian Assoc. for the Cultivation of Science, Vol. VI, part I, page 52.

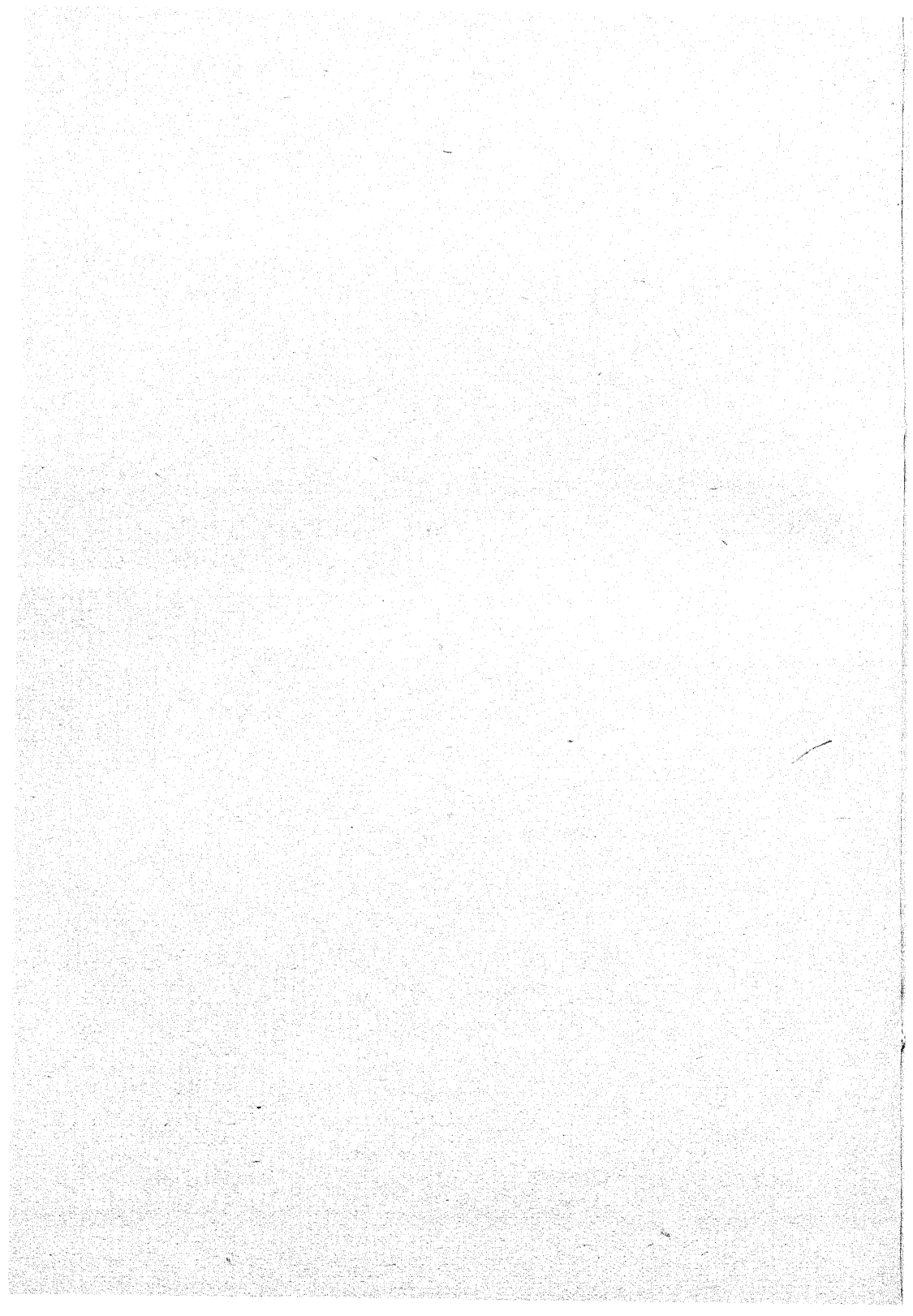
² It may be of interest to note here that since the return of Mr. R. N. Ghosh to Allahabad to join his appointment at the Ewing Christian College, the work with the ripple apparatus used by him has been continued at this laboratory. Mr. Goverdhan Lal Datta has shown that the apparatus can be used to exhibit the analogy of the effect of groove-form on grating spectra, and has succeeded in constructing a ripple grating which concentrates nearly all the energy in a single spectrum. Mr. Datta has also studied the character of interference and diffraction phenomena in the *immediate neighbourhood* of the sources which had not previously received adequate attention. Some very interesting results have also been obtained by Mr. J. C. Kameswara Rao on the forms of ripples of large amplitude. These are described in a paper appearing later in this volume. The work done has demonstrated the convenience and utility of the apparatus in the exact study of ripple motion. (C. V. R., 26 Oct., 1920.)

is running without interfering with the stroboscopic arrangement. The amplitude of the ripples excited can also be varied and kept strictly constant.

With the help of the apparatus the following new illustrations of optical theory by ripple motion have been worked out:—(1) Interferences near Caustics and the theory of the Rainbow; (2) Anamolous Propagation near Foci; (3) Unsymmetrical Types of Laminar Diffraction, and photographs of these cases are reproduced with the paper.

In conclusion, the author wishes to record his grateful acknowledgments to Prof. C. V. Raman for his suggestions and constant encouragement during the progress of this research.

DATED CALCUTTA, }
The 15th of June, 1920. }



XIII. The Theory of Impact on Elastic Plates.

By K. Seshagiri Rao, B.A. (Hons.), Research Scholar in the
Indian Association for the Cultivation of Science, Calcutta.

SYNOPSIS.

1. *Hertz's Theory*:—In a paper contributed to the *Physical Review* for April 1920, C. V. Raman has shown how Hertz's theory of the collision of elastic solids may be extended to the case of transverse impact on elastic plates so as to determine the proportion of the kinetic energy of the impinging body transformed to energy of wave motion in the plate. The assumptions forming the basis of this extension of Hertz's theory have been tested experimentally by the present author and it is shown how the residual discrepancies between observation and the results given by Raman's theory in the case of relatively thin plates may be explained.

2. *Duration of Impact*:—Except in the case of relatively thin plates the duration of impact is found to be substantially the same as that given by Hertz's theory of impact on an infinite mass of solid with a plane face; however, when the thickness of the plate is diminished to nearly the critical value at which the whole of the energy of the impinging body is transformed into energy of wave-motion, experiments show an appreciable increase in the duration of contact. It is shown that this is what might reasonably be expected on theoretical grounds.

3. *Apparent Coefficient of Restitution*:—Allowing for a small proportion of energy dissipated otherwise than in the production of wave motion in plate, and also making a correction for the increase of duration of impact referred to above, good agreement is found between theory and experiment for the values for the apparent coefficient of restitution. The increase of the coefficient of resti-

tution for decreasing velocity of impact is quantitatively explained.

4. *Acoustic Applications*:—The theory with suitable modifications appears to be capable of being applied to the problem of the analysis of the vibrations of bells or plates excited by impact.

I.—INTRODUCTION.

The mathematical theory of the collision of elastic solids given by Hertz is primarily applicable only to cases in which the shape of the impinging bodies and their velocity of collision is such that a negligible proportion of their kinetic energy is transformed into energy of wave motion in the solid or dissipated in producing quasi-permanent deformations. It is naturally of interest to determine how Hertz's theory has to be modified in various cases where such transformations of energy do occur to an appreciable extent. In a paper recently contributed to the *Physical Review* for April 1920, Prof. C. V. Raman has shown how the problem of transverse impact of a solid with curved faces on a plane elastic plate may be dealt with by an extension of the method suggested by Hertz, and has given a formula for the apparent coefficient of restitution of the impinging solid from which the proportion of its kinetic energy transformed into energy of wave motion in the plate may be calculated. In this application, it was assumed that the duration of impact was to a first approximation the same as that given by Hertz's theory for impact on the plane face of an infinite mass of solid. From the theory the result was deduced that as the thickness of the plate is gradually diminished, the apparent coefficient of restitution should continually diminish till when the thickness is below a certain critical limit depending upon the elastic constants and upon the mass, dimensions and velocity of the impinging body, the coefficient of restitution should vanish altogether: in other words that the body should then behave as if it were perfectly inelastic and remain in contact with the plate. It is obvious that when this critical limit is reached the duration of impact should become infinite: in other words that the very assumption on which theory itself is based should break down at or near this point. As a matter of fact, observation shows that while the calculated coefficient of restitution is in excellent agreement with the theory for plates of moderate thickness, deviations appear when the thick-

ness is rather near the critical limit and in such cases the observed coefficient of restitution falls less rapidly to zero with decreasing thickness of the plate than the calculated coefficient. It would appear from these considerations that in the case of the relatively thin plates the duration of impact should be somewhat larger than that given by Hertz's theory, and that this should be taken into account in the calculation of the apparent coefficient of restitution and that a better agreement might then be expected between theory and experiment in the case of such plates. The present paper describes work undertaken by the author mainly to clear up the outstanding discrepancies between theory and observation on the lines explained above.

II.—SOME EXPERIMENTAL RESULTS.

The materials selected for the investigation consisted of a set of five mild steel plates of thicknesses 3·82, 2·30, 1·81, 1·21 and 0·60 cms. respectively. The impinging body was a sphere of hard steel of diameter 2·70 cms.

The first set of experiments consisted in a determination of the duration of impact between sphere and plate for different velocities of impact. This was done in the usual way by allowing an electric circuit to be completed by the contact between the plate and the ball, the total quantity of electricity which passes from the time they meet till they again separate being measured on a ballistic galvanometer. Without taking into consideration the corrections for self-induction we get the formula

$$Q = \frac{V}{R} T \quad \text{or} \quad T = \frac{RQ}{V}$$

where R is the resistance, Q quantity of electricity and T the time of impact. The correction for self-induction are two in number.

(a) The current instead of rising instantaneously to its full value when the ball and the plate meet grows exponentially.

(b) When the ball separates from the plate a spark may be produced so that the current does not fall instantaneously to zero.

These corrections can be determined as follows: If a fairly large resistance be included in the circuit we can consider the contact resistance as negligible compared to the first. It follows therefore when the ball impinges the resistance rises to its final

value almost instantaneously. We may therefore treat the circuit as one having constant resistance. The equation of growth will then be the usual expression

$$L \frac{di}{dt} + Ri + V = 0$$

or

$$i = \frac{V}{R} \left(1 - e^{-\frac{R}{L}t} \right)$$

Integrating

$$Q = \frac{V}{R} \left\{ T - \frac{L}{R} \left(1 - e^{-\frac{R}{L}t} \right) \right\}$$

$$= \frac{V}{R} \left(T - \frac{L}{R} \right) \text{ if } \frac{R}{L} \text{ is large.}$$

$$\therefore T = \frac{RQ}{V} + \frac{L}{R}.$$

The correction is thus a constant addition to the time. The method of investigating these corrections was the same as used by Sears¹ for the problem of longitudinal impact of rods. The complete connections are shown in the figure.

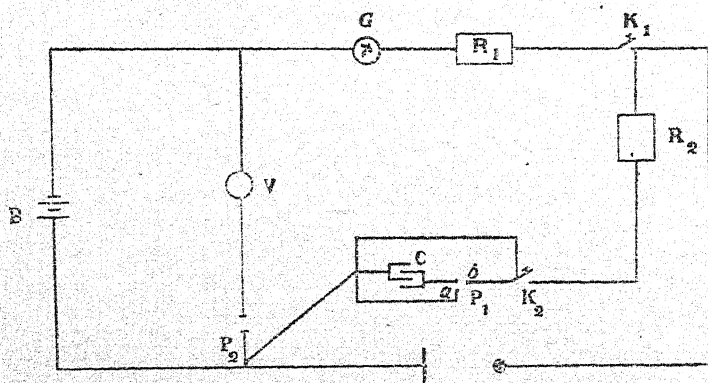


FIG. 1.

B was a battery of 4 volts and *R* was a resistance box from which 5000 ohms were unplugged. *C* was the condenser used in calibrating the galvanometer. With the high resistance used it was found that there was no sparking effect at all while the correction for self-induction was very small.

¹ Proceedings of Cambridge Philosophical Society, Vol. XIV, page 273.

The ball was suspended by a fine covered copper wire which was soldered to it, and which also served to convey the current during the impact. The distance of impact was regulated by holding the ball in position by an electro-magnet, care being taken that there was no leakage through the electro-magnet or through the suspension wire. Both the sphere and the surfaces of the plates on which the sphere impinged were highly polished and cleaned. It was found that concordant results could be obtained only when these were scrupulously clean, the slightest trace of dirt producing great variations in the readings.

For every plate the duration of impact was determined with different velocities. In every case it is found that the duration varies as the fifth root of velocity in the manner required by Hertz's theory. It is also found that the duration of contact is practically the same for the plates having thickness from 3·80 to 1·21 (though a very slight increase is to be noticed for decreasing thickness); but suddenly increases by 10% for the plate having a thickness of 0·6 cms. The table showing the results is given below.

TABLE I.

The height of suspension of the ball 74·35 cms.
Voltage applied 4·23.

Plate No. 1, Thickness 3·82 cm.

Withdrawal "	Galv. Throw θ	$\sqrt[5]{\frac{1}{\alpha \cdot \theta}}$
7·0 cm.	97·0 divisions.	157·8
4·2 "	108·0 "	157·2
2·3 "	129·0 "	158·5
2·2 "	136·0 "	158·1

Plate No. 2, Thickness 2.30 cms.

Withdrawal a	Galv. Throw θ	$\sqrt[5]{a \cdot \theta}$
7.3 cm.	105	156.3
5.9 „	110	157.0
4.1 „	119	157.5
2.7 „	130	158.0
1.5 „	146	158.0

Mean 157.6

Plate No. 3, Thickness 1.81 cm.

Withdrawal a	Galv. Throw θ	$\sqrt[5]{a \cdot \theta}$
7.8 cm.	104	156.7
6.4 „	109.5	159.0
5.3 „	114	159.0
3.6 „	122	157.7
2.5 „	132	158.3

Mean 158.2

Plate No. 4, Thickness 1.21 cms.

Withdrawal a	Galv. Throw θ	$\sqrt[5]{a \cdot \theta}$
7.3 cm.	106.0	158.8
3.9 „	119.0	164.8
3.1 „	129	161.4
2.2 „	135	157.8
3.3 „	128	162.4

Mean 160.6

Plate No. 5, Thickness 0.60 cm.

Withdrawal a	Galv. Throw θ	$\sqrt[5]{a} \cdot e$
7.6 cm.	118	177.0
6.3 „	121	174.0
5.1 „	126	174.0
3.7 „	134	175.8
2.7 „	142	174.5
1.5 „	162	175.4

Mean 175.1

It will be found that the readings in last column are practically constant for each plate and its change is little except for the thinnest plate.

If we take the mass of the plate to be very great and calculate the duration of contact on the basis of Hertz's theory, we find the calculated value agrees very well with the thicker plates while for the thinnest the discrepancy is about 10%.

TABLE II.

Velocity corresponding to 6 cms. withdrawal 21.78 cms./sec.

Calculated from Hertz's theory.	OBSERVED VALUES OF DURATION.				
	Plate No. 1.	No. 2.	No. 3.	No. 4.	No. 5.
1.265×10^{-4}	1.256×10^{-4}	1.256×10^{-4}	1.259×10^{-4}	1.281×10^{-4}	1.396×10^{-4}

The second set of experiments consisted in the determination of the coefficient of restitution for different velocities of impact. On Raman's theory the expression for e the coefficient of restitution is given by

$$e = \frac{f\rho a^2 - kM}{f\rho a^2 + kM}$$

where k a constant depending on the form of the transverse wave in the plate and

$$a^2 = \pi f \sqrt{E/3\rho(1-\sigma^2)}$$

where

E = Young's modulus,

τ = duration impact,

σ = Poison's ratio,

$2f$ = Thickness of plate,

ρ = density,

M = the mass of the impinging sphere.

It will be evident from the expression that e varies with τ , i.e. with the velocity of impact. e was determined experimentally for each plate for different impinging velocities and was calculated from the above expression, k being taken to be 0.56. τ , the duration of impact, is taken in the calculation for the first four plates to have given the value by Hertz's theory. For the thinnest plate, the calculated coefficient of restitution depends on whether τ is to be taken the actual value as found in experiment, or the value as given by Hertz's theory. Both the values are given, the former being marked by asterisks.

TABLE III.

Plate thickness.	Vel. 25.4		Vel. 21.8		Vel. 18.2		Vel. 14.5		Vel. 10.9	
	Calc.	Obs.	Calc.	Obs.	Calc.	Obs.	Calc.	Obs.	Calc.	Obs.
3.82 cm. ..	0.97	0.91	0.97	0.91	0.97	0.91	0.98	0.91	0.98	0.92
2.30 „ ..	0.93	0.87	0.94	0.88	0.94	0.88	0.94	0.89	0.94	0.89
1.81 „ ..	0.88	0.83	0.88	0.83	0.89	0.84	0.89	0.84	0.90	0.84
1.20 „ ..	0.77	0.70	0.77	0.71	0.78	0.72	0.79	0.72	0.79	0.73
0.60 „ ..	0.31*	0.30	0.32*	0.31	0.33*	0.32	0.35*	0.34	0.37*	0.36
	0.34		0.36		0.38		0.39		0.42	

In considering the degree of agreement of the calculated and observed coefficients of restitution shown in Table III it must be remembered that Raman's formula takes account of only the transformation of kinetic energy of translation into the energy of wave

motion in the plate and does not take into consideration the energy dissipated in production of permanent deformation in the plate. Even for the very moderate velocities of impact dealt with in the present investigation, some dissipation of the latter kind is evitable, distinct marks being left on the mild steel plates on the result of impact, and we may reasonably expect the observed coefficient of restitution for the thick as well as for the thin plates to be less than the calculated value by some units in the second place of decimals. This is exactly what appears in table III, in the case of the four moderately thick plates, and also for the thinnest plate if we take the asterisked values of the calculated coefficient for comparison with the experimental values.

DATED CALCUTTA, }
The 15th of May, 1920. }

XIV. On Ripples of Finite Amplitude.

By J. C. Kamesvara Rav, M.Sc., Palit Research Scholar
in the Calcutta University.

(Plate XII.)

CONTENTS.

- SECTION I.—Introduction.
SECTION II.—Description of Apparatus.
SECTION III.—Observation of the Form of Ripples of Large Amplitude.
SECTION IV.—Mathematical Theory for Plane Waves of Finite Amplitude.
SECTION V.—Mathematical Theory for Diverging Waves of Finite Amplitude.
SECTION VI.—Dependence of Velocity of Ripples on Amplitude and Depth of Liquid.
SECTION VII.—Outline of Further Research.
SECTION VIII.—Synopsis.

SECTION I.—INTRODUCTION.*

The problem of wave motion on the surface of liquids has for a long time been the subject of both theoretical and experimental study, its practical interest as well as the facility with which the various phenomena connected with it can be observed contributing to the development of the subject. Scott Russel¹ was one of the earlier observers in this field. Soon after Stokes² gave a complete theory of the oscillatory waves of finite amplitude and of permanent type. He found that to a second approximation, the velocity of waves does not depend upon the amplitude of the

* *Note.*—The experimental work described in this paper was carried out in June and July 1920. The mathematical theory was presented at a meeting of the Calcutta Mathematical Society on the 4th of September, 1920. The effects have since been observed also with mercury surfaces.

¹ Brit. Ass. Rep., Vol. VI, 1844.

² Camb. Trans. t. VIII (1847) and papers, Vol. I, p. 197.

waves, and as regards form he found that the hollows were no longer similar to the crests, but the height of the latter exceeds the depth of the former and that the crests are narrower than the hollows. For a third approximation he found that the velocity of propagation increases with the amplitude. In his investigations, he did not however take into account the effect of capillarity, but Kelvin¹ solved the problem completely for small amplitudes and proved the existence of a minimum velocity of propagation. Later on Rayleigh² studied ripples experimentally and observed them under intermittent illumination by the method of Foucault's test. By this method he measured the surface tensions of some liquids and the effect of contamination on the surface tension of water. Vincent³ studied ripples experimentally and took many beautiful photographs, illustrating interference, diffraction, etc., and more recently Watson⁴ used an improved method of Rayleigh's for the determination of surface tension and viscosity of liquids. Dr. Wilton⁵ gave a theory of the form of ripples of finite amplitude and short wave lengths and he showed mathematically that the length of the troughs should be very small compared with the length of the crests, and in fact the form of the troughs according to him have the shape of a cusp (ceratoid), where the curvature is very great. In the case of fairly large wave lengths (say 2.5 cm.) he obtained two forms theoretically, one similar to those of shorter ripples and the other with two crests in each wave length, and suggested (wrongly as it would appear) that the latter form was unstable.

It was with a view to test the accuracy of Wilton's work that the present investigation was first taken up, but in the course of the research, some quite new and interesting results were obtained. It was found that the form of ripples excited by a simple periodic pressure at a point were not simple undulations but with the increase of amplitude and wave length they assumed very complex forms varying with the conditions. The paper describes the effects observed and the mathematical theory is also worked out.

¹ Baltimore Lectures, p. 598; Phil. Mag. (4) t. XLVII, p. 374.

² Phil. Mag. (5) t. XXX, p. 386; Papers t. iii, p. 394.

³ Phil. Mag. Vol. 43, p. 417; 45, 191; 46, 290.

⁴ Phy. Rev. Vol. 12, p. 257 (1901); Phy. Rev. Vol. 15, p. 20 (1902); Phy. Rev. 2nd series, Vol. VII, p. 226 (1916).

⁵ Phil. Mag. May 1915, p. 688.

SECTION II.—DESCRIPTION OF THE APPARATUS.

The phenomena described in the present paper are best seen by the method of stroboscopic observation. It is possible though somewhat inconveniently to observe them when the ripples are excited in the usual manner by a low frequency fork, which is electrically maintained. A dipper is attached to one of the prongs and touches the surface of the water and when the fork is set in vibration it excites ripples on the surface. The tank containing water is a rectangular vessel of 5 cm. depth, with a glass bottom. To obtain intermittent illumination two slits are attached to the prongs of the fork, so that they allow light to pass through them once for every vibration and this intermittent light is passed through the bottom of the tank. This arrangement, however, gives an insufficient amount of light. In addition to this difficulty it is not possible to control and vary the amplitude of motion of the dipper and its frequency in the manner desirable in the present investigation. These difficulties are eliminated by the use of Professor Fleming's motor vibrator as improved by Professor C. V. Raman. The utility of this instrument for the study of vibrations cannot be overestimated.

The apparatus can be seen in fig. 1 of Plate XI. To the axle of the motor a circular wheel is fixed and to this wheel a brass disc carrying a slot and a movable pin is fixed. This pin actuates an oscillating lever, which moves vertically up and down. To the end of this lever a dipper is attached, which excites the ripples on the surface of the liquid contained in the tank. The tank also is shown in the figure. It is a rectangular vessel with a glass bottom.. To the other end of the axle of the motor a circular disc with a slot cut in it is attached. Light from an electric arc is allowed to pass through this hole and is reflected up by a mirror through the bottom of the tank and then by a second mirror projected on a screen. For every revolution of the motor, the light passes once, which makes the ripple pattern stationary. The dipper oscillating in the tank makes approximately a simple harmonic motion and to avoid any sideway motion it is curved into a circular arc of correct radius. In order to be sure that the observed phenomena are not due to any harmonics present in the vibration of the dipper, the experiment is also repeated with a dipper attached to the prong of an electrically maintained fork, the intermittent illumination being

obtained by allowing the beam of light to pass through the stroboscopic disk of the motor, the speed of the latter being suitably regulated.¹

It is also possible to observe the profile of the waves in the following manner. The intermittent flashes of light from the arc instead of being sent up through the bottom of the ripple tank are allowed to fall upon a white cardboard screen, the straight edge of which is viewed by the oblique reflection at the surface of the water contained in the tank. The edge appears in the form of a fixed wave which corresponds more or less closely with the actual wave-profile. The proper positions being chosen for the observer's eye and for the edge of the screen, satisfactory observations of the wave-profile may be made visually. The crests in the wave-profile correspond to the bright lines on the screen and the hollows to dark spaces.

SECTION III.—OBSERVATIONS OF THE FORM OF RIPPLES OF LARGE AMPLITUDE.

When the amplitude is small, we see on the screen for all wave-lengths a series of concentric circles at equal distances, corresponding to single crests in the waves, but when the amplitude is made great (say 2 mm.) the simple character of the waves is at once altered. For values of λ up to about 1.5 cm. the waves remain simple circles, but if the amplitude is made very great the first few waves divide. When λ lies between 1.5 and 3 cm. the circles become double, corresponding to two crests in each wave-length. For still greater wave-lengths the waves become triple and complex. At this stage it may be pointed out, that the word ripples is used in the popular sense and not in the sense used by Kelvin. Some photographs of the wave patterns are given in figs. 1 to 6, Plate XII. The figs. 1, 2, 3 correspond to a constant amplitude, but of increasing wave-length and figs. 4, 5, 6 correspond to a different amplitude and increasing wave-length. It will be seen from the figures that with the increase of wave-length the doubling of the crests becomes more prominent.

¹ The amplitude of the oscillator can be adjusted by altering the position of the pin in the brass slot and the wave length can be altered by altering the frequency of the motor, which can be managed by including a rheostat in the circuit of the motor.

Fig. 1.

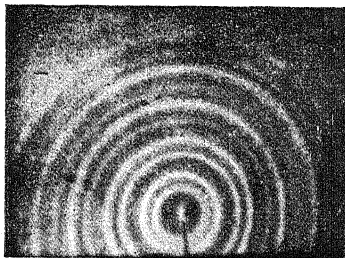


Fig. 4.

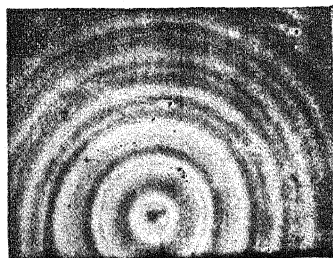


Fig. 2.

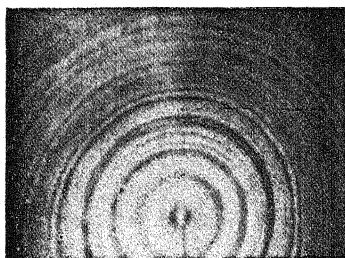


Fig. 5.

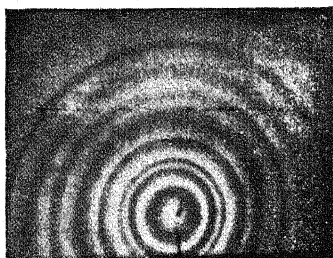


Fig. 3.

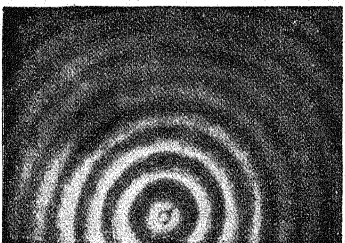
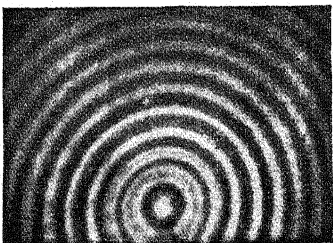
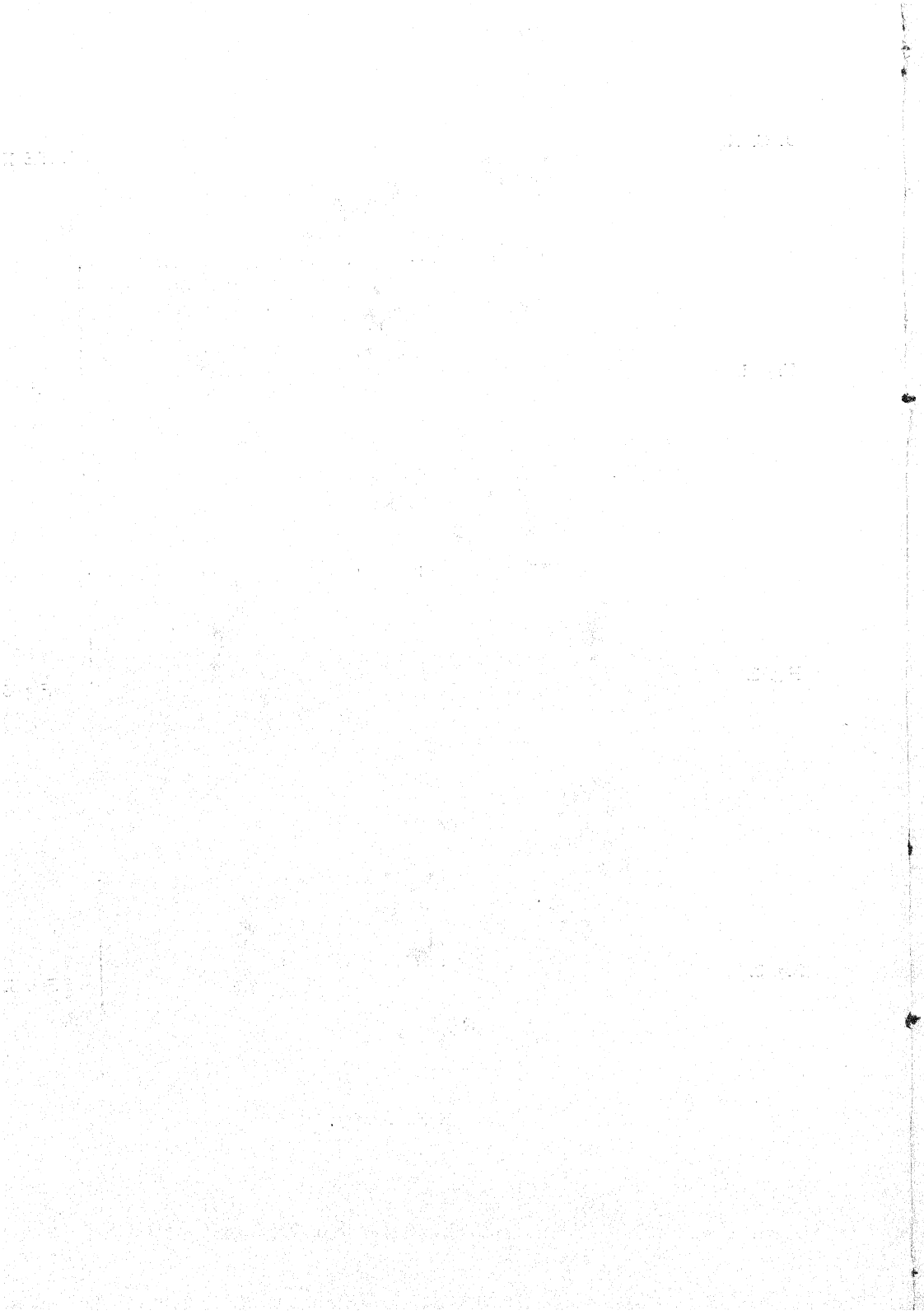


Fig. 6.



Ripples of Finite Amplitude.



Another feature that can be observed in these photographs is the change of form with increasing distance from the source. With increasing distance, the amplitude becomes smaller and consequently the circles become dim and in the case of the double pattern, one of the circles gradually becomes less prominent. This can be seen very well in figs. 3 and 6. Also, at a great distance the curvature becomes smaller, and consequently the bright, narrow circles near the centre broaden out slightly (figs. 1 and 4).

The change of the form of ripples with gradually decreasing depth of the liquid was also studied. Till the depth is reduced to a few millimeters no change is observed; the form then gradually becomes simpler. The critical depth at which the change of form occurs is, as might be expected, less for small wave-lengths and greater for the larger wave-lengths.

A curious anomalous effect is observed if the water be run out of the ripple tank while the dipper continues in oscillation; the wave-form then becomes simple, rather suddenly, and if we stop running out the water at that critical moment, we get sometimes simple crests and sometimes double crests. This effect is observed only when the amplitude is not very large, and is apparently connected with changes in the form of the contact surface between the dipper and the liquid. Usually, however, when the running out of the water from the tank is stopped, the form of the ripples corresponding to the particular depth is at once obtained.

We now proceed to explain the phenomena given in the previous section mathematically. First we take up the simpler case of plane waves.

SECTION IV.—MATHEMATICAL THEORY FOR PLANE WAVES OF FINITE AMPLITUDE.

Let the axis of x be in the direction of propagation of waves and the axis of y vertically upwards and the undisturbed level of the liquid as the plane $y=0$, and let the depth of the liquid be h . Then for irrotational two dimensional wave motion, the velocity potential satisfies the equation.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \dots\dots\dots (1)$$

$$\frac{\partial \phi}{\partial y} = 0 \text{ when } y = -h \quad \dots\dots\dots (2)$$

This condition denotes that the velocity at the bottom is zero and the pressure condition at the surface gives

$$-\frac{T}{\rho} \frac{\partial^2 y}{\partial x^2} = -gy + \frac{\partial \phi}{\partial t} - \frac{1}{2} \left\{ \left(\frac{\partial \phi}{\partial x} \right)^2 + \left(\frac{\partial \phi}{\partial y} \right)^2 \right\} \dots \dots \dots (3)$$

taking the curvature to be small, which is true for pretty long waves. Since the disturbing force is simple harmonic, we can suppose the following expression for ϕ

$$\phi = \sum_{s=1}^{s=n} a_s \cosh s m (y+h) \sin sm (ct-x) \dots \dots \dots (4)$$

which satisfies conditions (1) and (2).

We have also the condition

$$-\frac{\partial \phi}{\partial y} = \frac{\partial y}{\partial t}, \text{ and for } y=0$$

this gives

$$y = \frac{T}{c} \sum_{s=1}^{s=n} a_s \sinh smh \cos sm (ct-x) \dots \dots \dots (5)$$

To determine the constants $a_1, a_2, a_3 \dots$ we substitute the values of ϕ and y in the pressure equation (3) and equate the coefficients of $\cos sm (ct-x)$ to zero, we get equations for the determination of $c, a_2, a_3 \dots$ etc., a_1 itself being arbitrary. For practical purposes it is sufficient to take only three terms as the coefficients a_4, a_5 , etc., become negligibly small.

Thus we can take

$$\phi = a_1 \cosh m (y+h) \sin m (ct-x) + a_2 \cosh 2m (y+h) \sin 2m (ct-x) \\ + a_3 \cosh 3m (y+h) \sin 3m (ct-x)$$

and

$$y = \frac{T}{c} \left\{ a_1 \sinh mh \cos m (ct-x) + a_2 \sinh 2mh \cos 2m (ct-x) \right. \\ \left. + a_3 \sinh 3mh \cos 3m (ct-x) \right\}$$

Substituting these values of ϕ and y in the pressure equation (3) we get

$$\left(\frac{Tm^2}{\rho c} + \frac{g}{c} \right) a_1 \sinh mh \cos m (ct-x) + \left(\frac{4Tm^2}{\rho c} + \frac{g}{c} \right) a_2 \sinh 2mh \cos 2m (ct-x) \\ + \left(\frac{9Tm^2}{\rho c} + \frac{g}{c} \right) a_3 \sinh 3mh \cos 3m (ct-x) \\ = mc [a_1 \cosh m (y+h) \cos m (ct-x) + 2a_2 \cosh 2m (y+h) \cos 2m (ct-x) \\ + 3a_3 \cosh 3m (y+h) \cos 3m (ct-x)]$$

$$-\frac{m^2}{2} \left[\{a_1 \cosh m(y+h) \cos m(ct-x) + 2a_2 \cosh 2m(y+h) \cos 2m(ct-x) + 3a_3 \cosh 3m(y+h) \cos 3m(ct-x)\}^2 + \{a_1 \sinh m(y+h) \sin m(ct-x) + 2a_2 \sinh 2m(y+h) \sin 2m(ct-x) + 3a_3 \sinh 3m(y+h) \sin 3m(ct-x)\}^2 \right].$$

Expanding the right hand side of this equation in powers of y and substituting the value of y as given by equation (5) we get

$$\begin{aligned} & \left(\frac{Tm^2}{\rho c} + \frac{g}{c} \right) a_2 \sinh mh \cos m(ct-x) + \left(\frac{4Tm^2}{\rho c} + \frac{g}{c} \right) a_2 \sinh 2mh \cos 2m(ct-x) \\ & + \left(\frac{9Tm^2}{\rho c} + \frac{g}{c} \right) a_3 \sinh 3mh \cos 3m(ct-x). \\ & = mc \left[a_1 \cos m(ct-x) \left\{ 1 + \frac{m^2}{2!} \left(h + \sum_{s=1}^{s=3} \frac{a_s}{c} \sinh smh \cos sm(ct-x) \right)^2 \right. \right. \\ & \quad \left. \left. + \frac{m^4}{4!} \left(h + \sum_{s=1}^{s=3} \frac{a_s}{c} \sinh smh \cos sm(ct-x) \right)^4 + \dots \right\} \right. \\ & + 2a_2 \cos 2m(ct-x) \left\{ 1 + \frac{(2m)^2}{2!} \left(h + \sum_{s=1}^{s=3} \frac{a_s}{c} \sinh smh \cos sm(ct-x) \right)^2 \right. \\ & \quad \left. + \frac{(2m)^4}{4!} \left(h + \sum_{s=1}^{s=3} \frac{a_s}{c} \sinh smh \cos sm(ct-x) \right)^4 + \dots \right\} \\ & + 3a_3 \cos 3m(ct-x) \left\{ 1 + \frac{(3m)^2}{2!} \left(h + \sum_{s=1}^{s=3} \frac{a_s}{c} \sinh smh \cos sm(ct-x) \right)^2 \right. \\ & \quad \left. + \frac{(3m)^4}{4!} \left(h + \sum_{s=1}^{s=3} \frac{a_s}{c} \sinh smh \cos sm(ct-x) \right)^4 + \dots \right\} \Big] \\ & - \frac{m^2}{2} \left[a_1 \cos m(ct-x) \left\{ 1 + \frac{m^2}{2!} \left(h + \sum_{s=1}^{s=3} \frac{a_s}{c} \sinh smh \cos sm(ct-x) \right)^2 \right. \right. \\ & \quad \left. \left. + \frac{m^4}{4!} \left(h + \sum_{s=1}^{s=3} \frac{a_s}{c} \sinh smh \cos sm(ct-x) \right)^4 + \dots \right\} \right. \\ & + 2a_2 \cos 2m(ct-x) \left\{ 1 + \frac{(2m)^2}{2!} \left(h + \sum_{s=1}^{s=3} \frac{a_s}{c} \sinh smh \cos sm(ct-x) \right)^2 \right. \\ & \quad \left. + \frac{(2m)^4}{2!} \left(h + \sum_{s=1}^{s=3} \frac{a_s}{c} \sinh smh \cos sm(ct-x) \right)^4 + \dots \right\} \\ & + 3a_3 \cos 3m(ct-x) \left\{ 1 + \frac{(3m)^2}{2!} \left(h + \sum_{s=1}^{s=3} \frac{a_s}{c} \sinh smh \cos sm(ct-x) \right)^2 \right. \\ & \quad \left. + \frac{(3m)^4}{4!} \left(h + \sum_{s=1}^{s=3} \frac{a_s}{c} \sinh smh \cos sm(ct-x) \right)^4 + \dots \right\} \Big]^2 \end{aligned}$$

$$\begin{aligned}
& -\frac{m^2}{2} \left[a_1 \sin m (ct-x) \left\{ m \left(h + \sum_{s=1}^{s=3} \frac{a_s}{c} \sinh smh \cos sm (ct-x) \right) \right. \right. \\
& \quad \left. \left. + \frac{m^3}{3!} \left(h + \sum_{s=1}^{s=3} \frac{a_s}{c} \sinh smh \cos sm (ct-x) \right)^3 + \dots \right\} \right. \\
& \quad + 2a_2 \sin 2m (ct-x) \left\{ 2m \left(h + \sum_{s=1}^{s=3} \frac{a_s}{c} \sinh smh \cos sm (ct-x) \right) \right. \\
& \quad \left. + \frac{(2m)^3}{3!} \left(h + \sum_{s=1}^{s=3} \frac{a_s}{c} \sinh smh \cos sm (ct-x) \right)^3 + \dots \right\} \\
& \quad + 3a_3 \sin 3m (ct-x) \left\{ 3m \left(h + \sum_{s=1}^{s=3} \frac{a_s}{c} \sinh smh \cos sm (ct-x) \right) \right. \\
& \quad \left. \left. + \frac{(3m)^3}{3!} \left(h + \sum_{s=1}^{s=3} \frac{a_s}{c} \sinh smh \cos sm (ct-x) \right)^3 + \dots \right\} \right]
\end{aligned}$$

Equating the coefficients of $\cos m (ct-x)$, $\cos 2m (ct-x)$ and $\cos 3m (ct-x)$ to zero, in the above equation we get the following approximate equations for the determination of c , a_2 , and a_3 , a , itself being arbitrary.

$$\begin{aligned}
& \left(\frac{Tm^2}{\rho c} + \frac{g}{c} \right) \sinh mh - mc \cosh mh - \frac{3}{8} \frac{m^8}{c} a_1^2 \sinh^2 mh \cos mh \\
& - \frac{5}{2} m^2 a_2 \sinh mh \sinh 2mh + m^2 a_2 \cosh 3mh = 0 \dots \dots \dots (6)
\end{aligned}$$

$$\begin{aligned}
& \left(\frac{4Tm^2}{\rho c} + \frac{g}{c} \right) a_2 \sinh 2mh - 2mca_2 \cosh 2mh - 2 \frac{m^3}{c} a_1^2 a_2 \sinh^2 mh \cosh 2mh \\
& - \frac{m^8}{2c} a_1^2 a_2 \sinh mh \sinh 2mh \cosh 2mh + \frac{3}{2} \frac{m^8}{c} a_1^2 a_2 \sinh mh \sinh 3mh \\
& - \frac{m^8}{2} a_1^2 \sinh^3 mh + \frac{m^2}{4} - \frac{m^8}{c} a_2^3 \sinh^2 mh = 0 \dots \dots \dots (7)
\end{aligned}$$

$$\begin{aligned}
& \left(\frac{9Tm^2}{\rho c} + \frac{g}{c} \right) a_3 \sinh 3mh - 3mca_3 \cosh 3mh - \frac{5}{2} m^2 a_1 a_2 \sinh mh \sinh 2mh \\
& - \frac{m^8}{2c} a_1^2 a_3 \sinh^2 mh \sinh 3mh - \frac{27}{4} \frac{m^8}{c} a_1^2 a_3 \sinh^2 mh \cosh 3mh \\
& + \frac{3m^8}{c} a_1^2 a_3 \sinh mh \sinh 4mh - \frac{4m^2}{c} a_2^2 a_3 \sinh 2mh \sinh 3mh \cosh 2mh \\
& - \frac{27}{4} \frac{m^8}{c} a_2^2 a_3 \sinh^2 2mh \cosh 3mh + \frac{15}{2} \frac{m^3}{c} a_2^2 a_3 \sinh 5mh \sinh 2mh \\
& - 2a_2^2 a_1 \frac{m^3}{c} \cosh mh \sinh mh \sinh 2mh + \frac{3}{2} \frac{m^8}{c} a_1 a_2^2 \sinh mh \sinh 3mh \\
& + \frac{m^8}{c} a_1^2 a_3 \sinh 3mh \sinh 2mh + 4 \frac{m^8}{c} a_3^3 \sinh 4mh \sinh 3mh = 0 \dots \dots (8)
\end{aligned}$$

From equation (6) we get

$$c^2 = \left(\frac{Tm}{\rho} + \frac{g}{m} \right) \tanh mh - \frac{3}{8} \frac{m^8}{c} a_1^2 \sinh^2 mh - \frac{5}{2} m^2 a_2 \tanh mh \sinh 2mh \\ + m^2 a_2 \frac{\cosh 3mh}{\cosh mh} \dots \dots \dots (9)$$

From equation (7) we get

$$a_2 = \frac{\frac{m^2}{2} a_1^2 \sinh^2 mh - \frac{m^2}{4}}{\left(\frac{4Tm^2}{\rho c} + \frac{g}{c} \right) \sinh 2mh - 2mc \cosh 2mh - \frac{2m^8}{c} a_1^2 \sinh^2 mh \cosh 2mh \\ - \frac{m^8}{2c} a_1^2 \sinh mh \sinh 2mh \cosh mh + \frac{3}{2} \frac{m^8}{c} a_1^2 \sinh mh \sinh 3mh. \\ \dots \dots \dots (10)}$$

From equation (8) we have

$$a_3 = \frac{\frac{5}{2} m^2 a_1 a_2 \sinh mh \sinh 2mh + a_1 a_2^2 \frac{m^8}{c} \sinh^2 2mh - \frac{3}{2} \frac{m^8}{c} a_1 a_2^2 \sinh mh \sinh 3mh}{\left(\frac{9Tm^2}{\rho c} + \frac{g}{c} \right) \sinh 3mh - 3mc \cosh 3mh - \frac{m^8}{2c} a_1^2 \sinh^2 mh \sinh 3mh \\ - \frac{27}{4} \frac{m^8}{c} a_1^2 \sinh^2 mh \cosh 3mh \\ + \frac{3m^8}{c} a_1^2 \sinh mh \sinh 4mh - \frac{4m^8}{c} a_2^2 \sinh 2mh \sinh 3mh \cosh 2mh \\ - \frac{27}{4} \frac{m^8}{c} a_2^2 \sinh^2 2mh \cosh 3mh + \frac{15}{2} m^8 a_2^2 \sinh 5mh \sinh 2mh. \\ \dots \dots \dots (11)}$$

Hence

$$y = \frac{a_1}{c} \sinh mh \cos m (ct - x) \\ + \frac{\left(\frac{m^2}{2} a_1^2 \sinh^2 mh - \frac{m^2}{4} \right) \sinh 2mh \cos 2m (ct - x)}{\left(\frac{4Tm^2}{\rho} + g \right) \sin 2mh - 2mc^2 \cosh 2mh - m^8 a_1^2 \left(2 \sinh^2 mh \cosh 2mh + \right. \\ \left. \dots + \frac{1}{4} \sinh^2 2mh - \frac{3}{2} \sinh mh \sinh 3mh \right)}$$

$$\begin{aligned}
& \frac{3}{2} m^2 a_1 a_2 \sinh mh \sinh 2mh + a_1 a_2^2 \frac{m^3}{c} \sinh^2 2mh \\
& - \frac{3}{2} \frac{m^3}{c} a_1 a_2^2 \sinh mh \sinh 3mh \\
& + \frac{\left(\frac{9Tm^2}{\rho} + g \right) \sinh 3mh \frac{3}{8} mc^2 \cosh 3mh - m^3 a_1^2}{\left(\frac{1}{2} a_1^2 \sinh^2 mh \sinh 3mh + \frac{27}{4} + \dots \right)} \\
& - m^2 a_2^2 (\dots)
\end{aligned}$$

Putting $\frac{a_1}{c} \sinh mh = a$, the above formula becomes

$$\begin{aligned}
y = a \cos m(ct-x) + & \frac{\left(\frac{c^2 m^2}{2} - \frac{m^2 c^2}{4 \sinh^2 mh} \right) \sinh 2mh \cos 2m(ct-x)}{\left(\frac{4Tm^2}{\rho} + g \right) \sinh 2mh - 2mc^2 \cosh 2mh} \\
& - m^3 a^2 \left(2 \cosh 2mh + \cosh^2 mh - \frac{3}{2} \frac{\cosh 3mh}{\cosh mh} \right) \\
& \left\{ \frac{5}{2} cm^2 a a_2 \sinh 2mh + a a_2^2 m^3 \left(\frac{\sinh^3 2mh}{\sinh mh} - \frac{3}{2} \sinh 3mh \right) \right\} \\
& \times \sinh 3mh \cos 3m(ct-x) \\
& + \frac{\left(\frac{9Tm^2}{\rho} + g \right) \sinh 3mh - 3mc^2 \cosh 3mh - c^2 m^3 a^2 \left(\frac{1}{2} \sinh 3mh \right.}{\left. + \frac{27}{4} \cosh 3mh - \frac{3}{2} \frac{\sinh 4mh}{\sinh mh} \right)} \\
& - m^3 a_2^2 \left(2 \sinh 4mh \sinh 3mh + \frac{27}{4} \sinh^2 mh \cosh 3mh \right. \\
& \left. - \frac{15}{2} \sinh 5mh \sinh 2mh \right) \\
& \dots \dots \dots (12)
\end{aligned}$$

For infinite depth this formula becomes

$$\begin{aligned}
y = a \cos m(ct-x) + & \frac{\frac{m^2}{2} a^2 c^2 \cos 2m(ct-x)}{\left(\frac{4Tm^2}{\rho} + g \right) - 2mc^2 + \frac{1}{2} m^3 a^2 c^2 - m^3 a'^2 c^2} \\
& + \frac{\frac{5}{2} c^2 m^2 a a' - \frac{1}{2} c^2 m^3 a a'^2}{\left(\frac{9Tm^2}{\rho} + g \right) - 3mc^2 + \frac{3}{4} m^3 c^2 a^2 + \frac{49}{4} m^3 a'^2 c^2} \\
& \dots \dots \dots (12a)
\end{aligned}$$

where
$$a' = \frac{\frac{m^2}{2}a^2}{\left(\frac{4Tm^2}{\rho} + g\right) - 2mc^2 + \frac{1}{2}m^3a^2c^2},$$

and
$$c^2 = \frac{Tm}{\rho} + \frac{g}{m}.$$

From this formula, the curves given below have been constructed to show the change of form as we increase the wave length.

In Fig. 7, $a = .2$ and $m = 3$

$$y = .2 \cos m (ct - x) + .16 \cos 2m (ct - x) + .03 \cos 3m (ct - x)$$

In Fig. 8, $a = .2$ and $m = 2.5$

$$y = .2 \cos m (ct - x) - .12 \cos 2m (ct - x) - .001 \cos 3m (ct - x)$$

In Fig. 9, $a = .2$ and $m = 2.25$

$$y = .2 \cos m (ct - x) - .02 \cos 2m (ct - x) - .07 \cos 3m (ct - x)$$

In Fig. 10, $a = .2$ and $m = 2$.

$$y = .2 \cos m (ct - x) - .1 \cos 2m (ct - x) - .05 \cos 3m (ct - x).$$



FIG. 7.



FIG. 8.



FIG. 9.



FIG. 10.

The figures given in figures 7 to 10 correspond closely with those obtained experimentally for the same amplitudes and wave lengths. For practical purposes a depth of 4 or 5 cms. serves our purpose. The formula given above shows how these changes of

form come only between certain wave lengths, and how far large gravity waves, a_2 and a_3 become small and then the form becomes cycloidal,¹ as obtained by Stokes. The formula given above gives the form of small ripples as well as large gravity waves.

The formula (12) shows also the change of form of the waves with depth. From the numerator of the coefficient of $\cos 2m(ct-x)$ we can see how the change is more prominent for small values of m , i.e. for large values of λ .

We shall next discuss the case of the diverging waves mathematically.

SECTION V.—MATHEMATICAL THEORY FOR DIVERGING WAVES.

In this case we have to take cylindrical co-ordinates and take the axis of z vertically upwards and as before the undisturbed surface of the water as the plane, $z=0$.

The equations are modified as

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = 0 \dots \dots \dots (13)$$

$$\frac{\partial \phi}{\partial r} = 0 \text{ when } r = \infty \dots \dots \dots (14)$$

$$\frac{\partial \phi}{\partial z} = 0 \text{ when } z = -h \dots \dots \dots (15)$$

The pressure equation becomes

$$-\frac{T}{\rho} \frac{\partial^2 \phi}{\partial r^2} = -gz + \frac{\partial \phi}{\partial t} - \frac{1}{2} \left\{ \left(\frac{\partial \phi}{\partial z} \right)^2 + \left(\frac{\partial \phi}{\partial r} \right)^2 \right\} \dots \dots \dots (16)$$

We can therefore assume

$$\phi = \sum_{s=1}^{s=n} a_s J_0(skr) \cosh sk(z+h) \sin skct \text{ where } k = \frac{2\pi}{\lambda} \dots \dots \dots (17)$$

and from the condition $-\frac{\partial \phi}{\partial z} = \frac{\partial z}{\partial t}$ we get for $z=0$

$$z = \frac{1}{c} \sum_{s=1}^{s=n} a_s J_0(skr) \sinh shk \cos skct \dots \dots \dots (18)$$

Substituting these values in the pressure equation (16), and taking only three terms into consideration as in the case of plane waves we get

¹ For large gravity waves the formula reduces to

$$y = a \cos m(ct-x) - \frac{1}{2} ma^2 \cos 2m(ct-x) + \frac{5}{8} m^2 a^3 \cos 3m(ct-x).$$

$$\begin{aligned}
& -\frac{Tk^2}{\rho c} \left[\sum_{s=1}^{s=3} s^2 a_s J_0(skr) \sinh shk \cos skt \right] \\
& = -\frac{g}{c} \sum_{s=1}^{s=3} a_s J_0(skr) \sinh shk \cos skt + kc \sum_{s=1}^{s=3} sa_s J'_0(kr) \cosh sk(z+h) \sin skt \\
& \quad - \frac{k^2}{2} \left\{ \left[\sum_{s=1}^{s=3} sa_s J_0(kr) \sinh sk(z+h) \sin skt \right]^2 \right. \\
& \quad \left. + \left[\sum_{s=1}^{s=3} sa_s J'_0(kr) \cosh sk(z+h) \cos skt \right]^2 \right\}
\end{aligned}$$

Expanding the right hand side of this equation in series of z and substituting the value of z given by (18) we get

$$\begin{aligned}
& \sum_{s=1}^{s=3} \left\{ -\frac{s^2 Tk^2}{\rho c} J_0''(skr) + \frac{g}{c} J_0(skr) \right\} a_s \sinh shk \cos skt \\
& = kc a_1 J_0(kr) \cos kct \left[1 + \frac{k^2}{2!} \left(h + \sum_{s=1}^{s=3} \frac{a_s}{c} J_0(skr) \sinh shk \cos skt \right)^2 \right. \\
& \quad \left. + \frac{k^4}{4!} \left(h + \sum_{s=1}^{s=3} \frac{a_s}{c} J_0(skr) \sinh shk \cos skt \right)^4 + \dots \right] \\
& + 2kc J_0(2kr) \cos 2kct \left[1 + \frac{(2k)^2}{2!} \left(h + \sum_{s=1}^{s=3} \frac{a_s}{c} J_0(skr) \sinh shk \cos skt \right)^2 \right. \\
& \quad \left. + \frac{(2k)^4}{4!} \left(h + \sum_{s=1}^{s=3} \frac{a_s}{c} J_0(skr) \sinh shk \cos skt \right)^4 + \dots \right] \\
& + 3kc J_0(3kr) \cos 3kct \left[1 + \frac{(3k)^2}{2!} \left(h + \sum_{s=1}^{s=3} \frac{a_s}{c} J_0(skr) \sinh shk \cos skt \right)^2 \right. \\
& \quad \left. + \frac{(3k)^4}{4!} \left(h + \sum_{s=1}^{s=3} \frac{a_s}{c} J_0(skr) \sinh shk \cos skt \right)^4 + \dots \right] \\
& - \frac{k^2}{2} \left[a_1 J'_0(kr) \sin kct \right] \left\{ 1 + \frac{k^2}{2} \left(h + \sum_{s=1}^{s=3} \frac{a_s}{c} J_0(skr) \sinh shk \cos skt \right)^2 \right. \\
& \quad \left. + \frac{k^4}{4!} \left(h + \sum_{s=1}^{s=3} \frac{a_s}{c} J_0(skr) \sinh shk \cos skt \right)^4 + \dots \right\} \\
& + 2a_2 J'_0(2kr) \sin 2kct \left\{ 1 + \frac{(2k)^2}{2!} \left(h + \sum_{s=1}^{s=3} \frac{a_s}{c} J_0(skr) \sinh shk \cos skt \right)^2 \right. \\
& \quad \left. + \frac{(2k)^4}{4!} \left(h + \sum_{s=1}^{s=3} \frac{a_s}{c} J_0(skr) \sinh shk \cos skt \right)^4 + \dots \right\} \\
& + 3a_3 J'_0(3kr) \sin 3kct \left\{ 1 + \frac{(3k)^2}{2!} \left(h + \sum_{s=1}^{s=3} \frac{a_s}{c} J_0(skr) \sinh shk \cos skt \right)^2 \right. \\
& \quad \left. + \frac{(3k)^4}{4!} \left(h + \sum_{s=1}^{s=3} \frac{a_s}{c} J_0(skr) \sinh shk \cos skt \right)^4 + \dots \right\} \Bigg]^2
\end{aligned}$$

$$\begin{aligned}
& -\frac{k^2}{2} \left[a_1 J_0(kr) \sin kct \left\{ k \left(h + \sum_{s=1}^{s=3} \frac{a_s}{c} J_0(skr) \sinh s kh \cos s kct \right) \right. \right. \\
& \quad \left. \left. + \frac{k^3}{3!} \left(h + \sum_{s=1}^{s=3} \frac{a_s}{c} J_0(skr) \sinh s kh \cos s kct \right)^3 + \dots \right\} \right. \\
& + 2a_2 J_0(2kr) \sin 2kct \left\{ 2k \left(h + \sum_{s=1}^{s=3} \frac{a_s}{c} J_0(skr) \sinh s kh \cos s kct \right) \right. \\
& \quad \left. + \frac{(2k)^3}{3!} \left(h + \sum_{s=1}^{s=3} \frac{a_s}{c} J_0(skr) \sinh s kh \cos s kct \right)^2 + \dots \right\} \\
& + 3a_3 J_0(3kr) \sin 3kct \left\{ 3k \left(h + \sum_{s=1}^{s=3} \frac{a_s}{c} J_0(skr) \sinh s kh \cos s kct \right) \right. \\
& \quad \left. + \frac{(3k)^3}{3!} \left(h + \sum_{s=1}^{s=3} \frac{a_s}{c} J_0(skr) \sinh s kh \cos s kct \right)^3 + \dots \right\} \Big]^2
\end{aligned}$$

Equating the co-efficients of $\cos kct$, $\cos 2kct$ and $\cos 3kct$ to zero, we get the following approximate equations for the determination of c , a_2 and a_3 .

$$\begin{aligned}
& -\frac{Tk^2}{\rho c} J_0''(kr) \sinh kh + \frac{g}{c} J_0(kr) \cosh kh - kc J_0(kr) \cosh kh \\
& -\frac{k^2}{2c} a_1^2 \{J_0(kr)\}^2 \sinh^2 kh - \frac{3}{2} \frac{k^2}{c} a_2 J_0(kr) J_0(2kr) \sinh mh \sinh 2mh \\
& + \frac{k^2}{2} a_2 J_1(kr) J_1(2kr) \cosh mh \cosh 2mh = 0 \dots\dots\dots (19)
\end{aligned}$$

$$\begin{aligned}
& -\frac{4Tk^2}{\rho c} a_2 J_0''(2kr) \sinh 2kh + \frac{g}{c} a_2 J_0(2kr) \sinh 2kh \\
& -2kc a_2 J_0(2kr) \cosh 2kh - \frac{3}{4} k^2 a_1^2 \{J_0(kr)\}^2 \sinh^2 kh \\
& + \frac{k^2}{4} a_1^2 \{J_1(kr)\}^2 \cosh^2 kh = 0 \dots\dots\dots (20)
\end{aligned}$$

$$\begin{aligned}
& -\frac{9Tk^2}{\rho c} a_3 J_0''(3kr) \sinh 3kh + \frac{g}{c} a_3 J_0(3kr) \sinh 3kh \\
& -3kc a_3 J_0(3kr) \cosh 3kh - \frac{3}{2} a_1 a_2 J_0(kr) J_0(2kr) \sinh kh \sinh 2kh \\
& + \text{etc.} \dots = 0 \dots\dots\dots (21)
\end{aligned}$$

From (19) we have

$$\begin{aligned}
c^2 = & -\frac{Tk}{\rho} \cdot \frac{J_0''(kr)}{J_0(kr)} \tanh kh + \frac{g}{k} \tanh kh - \frac{k}{2} a_1^2 \{J_0(kr)\}^2 \sinh kh \tanh kh \\
& + k^2 a_2 \frac{J_1(kr)}{J_0(kr)} J_1(2kr) \cosh kh \cosh 2kh - \frac{3}{2} \frac{k^2}{c} a_2 J_0(2kr) \tanh kh \sinh 2mh.
\end{aligned}$$

$$\text{Since } J_0''(kr) = -J_0(kr) + \frac{1}{kr} J_1(kr)$$

We have as a first approximation

$$c^2 = \left(-\frac{Tk}{\rho} \frac{J_0''(kr)}{J_0(kr)} + \frac{g}{k} \right) \tanh kh$$

$$= \left(\frac{Tk}{\rho} - \frac{Tk}{\rho} \cdot \frac{1}{kr} \frac{J_1(kr)}{J_0(kr)} + \frac{g}{k} \right) \tanh kh.$$

From (20) we have

$$a_2 = \frac{\frac{3}{4} k^2 a_1^2 \{J_0(kr)\}^2 \sinh^2 kh + \frac{k^2 a_1^2}{4} \{J_1(kr)\}^2 \cosh^2 kh}{- \frac{4Tk^2}{\rho c} J_0''(2kr) \sinh 2kh + \frac{g}{c} J_0(2kr) \sinh 2kh - 2kc J_0(2kr) \cosh 2kh}.$$

From (21) we have

$$a_3 = \frac{\frac{3}{2} a_1 a_2 J_0(kr) J_0(2kr) \sin kh \sinh 2kh - a_1 a_2 J_1(kr) (J_2 kr) \cosh kh \cosh 2kh}{- \frac{9Tk^2}{\rho c} J_0''(3kr) \sinh 3kh + \frac{g}{c} J_0(3kr) \sinh 3kh - 3 J_0(3kr) \cosh 3kh - \frac{3}{2} a_1 a_2 J_0(kr) J_0(2kr) \sinh kh \sinh 2kh + \dots}$$

Hence

$$z = \frac{a_1}{c} \sinh kh J_0(kr) \cos kct$$

$$+ \frac{1}{c} \frac{\left[\frac{3}{4} k^2 a_1^2 \{J_0(kr)\}^2 \sinh^2 kh + \frac{k^2 a_1^2}{4} \{J_1(kr)\}^2 \cosh^2 kh \right] \sinh 2kh J_0(2kr) \cos 2kct}{- \frac{4Tk^2}{\rho c} J_0''(2kr) \sinh 2kh + \frac{g}{c} J_0(2kr) \sinh 2kh - 2kc J_0(2kr) \cosh 2kh}$$

$$+ \frac{1}{c} \frac{\frac{3}{2} a_1 a_2 J_0(kr) J_0(2kr) \sinh kh \sinh 2kh - a_1 a_2 J_1(kr) J_1(2kr) \cosh kh \cosh 2kh}{- \frac{9Tk^2}{\rho c} J_0''(3kr) \sinh 3kh + \frac{g}{c} J_0(3kr) \sinh 3kh - 3kc J_0(3kr) \cosh 3kh - \frac{3}{2} a_1 a_2 (\dots) \times \sinh 3kh J_0(3kr) \cos 3kct}$$

Putting $\frac{a_1}{c} \sinh kh = a$, and simplifying we get

$$z = a J_0(kr) \cos kct$$

$$+ \frac{\left[\frac{3}{4} c^2 k^2 a^2 \{J_0(kr)\}^2 + \frac{k^2}{4} c^2 a^2 \{J_1(kr)\}^2 \coth^2 kh \right] \sinh 2kh \cos 2kct}{- \frac{4Tk^2}{\rho} \frac{J_0''(2kr)}{J_0(2kr)} \sinh 2kh + g \sinh 2kh - 2kc^2 \cosh 2kh}$$

$$\frac{3}{2} ca a_2 J_0(kr) J_0(2kr) \sinh 2kh - ca a_2 J_1(kr) J_1(2kr) \coth kh \cosh 2kh \\ + \frac{\times \sin 3kh J_0(3kr) \cos 3kct}{- \frac{9Tk^2}{\rho} J_0''(3kr) \sinh 3kh + g J_0(3kr) \sinh 3kh - 3c^2k J_0(3kr) \cosh 3kh}$$

For infinite depth this formula reduces to

$$z = a J_0(kr) \cos kct + \frac{\left[\frac{3}{4} c^2 k^2 a^2 \{J_0(kr)\}^2 + \frac{c^2 k^2 a^2}{4} \{J_1(kr)\}^2 \right] \cos 2kct}{\left(-\frac{4Tk^2}{\rho} \frac{J_0''(2kr)}{J_0(2kr)} + \frac{g}{c} \right) - 2kc^2 + \dots} \\ + \frac{\frac{3}{2} ca a_2 J_0(kr) J_0(2kr) - ca a_2 J_1(kr) J_1(2kr)}{-\frac{9Tk^2}{\rho} \frac{J_0''(3kr)}{J_0(3kr)} + g - 3kc^2 - \frac{3}{2} a (\dots)} \cos 3kct.$$

The asymptotic values of $J_0(kr)$ is $\sqrt{\frac{2}{\pi kr}} \cos\left(kr - \frac{\pi}{4}\right)$, and that of $J_1(kr)$ is $\sqrt{\frac{2}{\pi kr}} \sin\left(kr - \frac{\pi}{4}\right)$

Substituting these values in the above formula, we get

$$z = a \sqrt{\frac{2}{\pi kr}} \cos\left(kr - \frac{\pi}{4}\right) \cos kct + \frac{2}{\pi kr} \cdot \frac{\frac{1}{4} k^2 a^2 (1 + 2 \sin 2kr)}{-\frac{4Tk^2}{\rho} \frac{J_0''(2kr)}{J_0(2kr)} + g - 2kc^2} \cos 2kct \\ + \text{etc.}$$

The above formula shows that the amplitude of the second harmonic decreases inversely as the distance from the origin, whereas the fundamental decreases inversely as the square root of the distance. Hence the wave form would tend to become simpler with the increase of distance from the source. This is what is also found experimentally.

SECTION VI.—DEPENDENCE OF THE VELOCITY OF RIPPLES ON AMPLITUDE AND DEPTH OF LIQUID.

$$c^2 = \left(\frac{Tm}{\rho} + \frac{g}{m} \right) (1 + am^2 a^2).$$

In the case of waves of finite amplitude, a becomes a positive fraction. The value of a can also be easily deduced¹ for large waves

¹ Lamb's Hydrodynamics (4th edition), p. 410.

by taking only one term in the series and by using the method of successive approximation.

In order to find the velocity of propagation experimentally, it is necessary to find the absolute frequency of the motor. This can be easily measured by means of an electric tachometer. But for purposes of comparison it is not necessary to find the absolute frequency. Since the velocity is proportional to wave-length, if the frequency remains constant, it will be sufficient to measure only the wave-lengths. The amplitudes of oscillation are measured by means of a travelling microscope or simply by means of the scale in the eye piece of a low power microscope. The wave lengths are measured on the screen and knowing the ratio of the length of a line on the surface of water and its image on the screen we can find the absolute value of the wave length of the ripples. In taking these measurements, it is absolutely necessary to keep the frequency of the motor constant, or otherwise the results are not comparable. This can be easily secured by means of the rheostat which is included in the circuit of the motor. The frequency can be read off directly from the electric tachometer. Some observations made in this manner show that the velocity of long waves increases with the amplitude as indicated by the theory. A more detailed comparison between theory and experiment is at present being carried out.

Neglecting for the time being the effect of amplitude we get

$$c^2 = \left(\frac{Tm}{\rho} + \frac{g}{m} \right) \tanh mh$$

This form is well known and it is sufficient here simply to indicate that this value of c^2 agrees with the experimental results when the amplitude is very small. From the formula it can be seen that for small wave lengths the velocity continues to be constant till a very low depth of a few millimeters is reached. But for large wave lengths, i.e. for small values of m , the increase in the value of c^2 becomes apparent even for greater depths. This result is in agreement with what is actually observed.

SECTION VII.—OUTLINE OF FURTHER RESEARCH TO BE CARRIED OUT.

The preceding section describes the work that has been carried out so far. There is much that still remains to be done. In the

present investigation regarding the form of ripples, we have not taken into account the effect of viscosity. The chief effect of viscosity is to diminish the amplitude and this in its turn affects the form of the wave. It appears also likely that the variation of the velocity with amplitude may have a marked effect on the group velocity of ripples of finite amplitude. It will be interesting to investigate this experimentally. In the theory of group velocity usually given, the effect of finite amplitude on the speed of the waves is not taken into account, and it seems possible that this may considerably influence the results. As we have seen, the effect of increasing the amplitude beyond a certain point is to cause the waves to divide and it would be interesting to make direct measurements of the amplitude of the waves in such cases and to compare the same with theory. The dependence of ripple form on depth, especially in those cases in which the tank is of variable depth also appears to merit fuller investigation. The vessel in which the liquid is contained might, for instance, be wedge-shaped or hemispherical, and so on. Experiments with other liquids might also be carried out.

SECTION VIII.—SYNOPSIS.

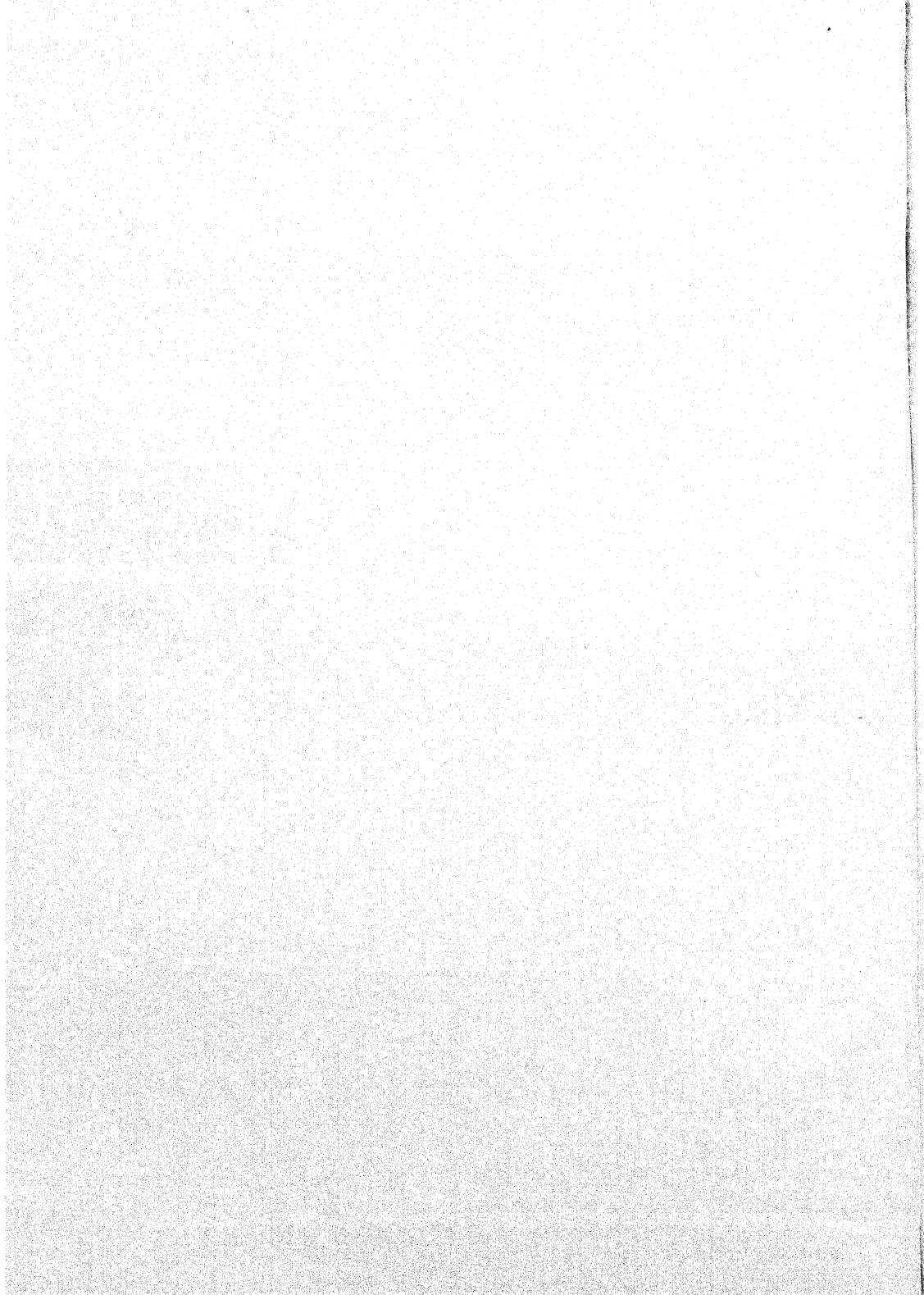
It was shown in the previous sections how the form of ripples of different wave lengths changes with the amplitude and frequency. To study these forms Fleming's motor vibrator as improved by Prof. Raman was used to excite the ripples. This vibrator has many advantages over the ordinary electrically maintained tuning fork, viz. the amplitude of vibration and the frequency can be regulated to a nicety. The bottom of the tank is made of a glass plate and is illuminated by intermittent light, so that the ripples appear stationary.

It was found that when the amplitude is small, the ripples were simple undulations for all wave lengths, but for finite amplitudes, with the increase of wave length, the crests of the waves become double, triple, and so on. With increasing distance from the source, these forms become simpler, as also with decreasing depth of the liquid in the tank. A mathematical theory explaining all these phenomena is given, both for plane and diverging waves.

It is also shown that the velocity of large waves increases with amplitude and the mathematical results are at present being

verified. The way in which the velocity depends upon the depth is also worked out, and there it is shown that the velocity decreases with the depth, though the change is not perceptible in the case of smaller waves till a very smaller depth is reached.

In conclusion, the author wishes to express his best thanks to Prof. C. V. Raman who suggested the investigation for the facilities afforded and for his constant encouragement and interest during the progress of the research.



XV. The Effects of a Magnetic Field on the General Spectrum.

By H. P. Waran, M.A., Government Scholar of the
University of Madras.

(Plate XIII, Fig. 2.)

The complex change taking place in the source when the radiation is emitted in a magnetic field does not seem to have had the attention it deserves. Though the spectrum shows a variety of changes under the influence of the field, yet the study of the spectrum has mainly been confined to the Zeeman effect on a particular line isolated from the rest of the spectrum.

Some remarkable changes observed in the case of a mercury discharge-tube brought the phenomenon to my notice and some aspects of the problem with the rare gases were experimentally worked out and some of the very interesting results obtained summarised in a paper¹ read before the Cambridge Philosophical Society.

Since then my attention has been drawn to the fact that some work has already been done on the subject but without coming to any definite conclusions. As early as 1870 Rand Capron in his book on Aurora mentions some experiments with a discharge-tube placed in a magnetic field, and speaks of some lines being brought out by it.

The latest paper² on the question has been by Norton A. Kent and Royal M. Frye, where a bibliography of the past work on the subject is also given. In this paper though the authors set out with the idea of elucidating the real nature of the phenomenon,

¹ Proceedings, Cambridge Philosophical Society, 1920.

² *Astrophysical Journal*, Vol. 37 (1913), p. 183.

the experimental conditions they adopted seem to have led them to doubtful conclusions. In the main they attribute the changes observed to the disruptive action of the discharge when it is deflected by the field into the walls of the tube, the changes in the spectrum being due to the products of dissociation of the glass. Though this conclusion is fairly justifiable from their experiments, yet there are a few aspects of the problem which are not so easily explained. In fact my observations point to the contrary effect,

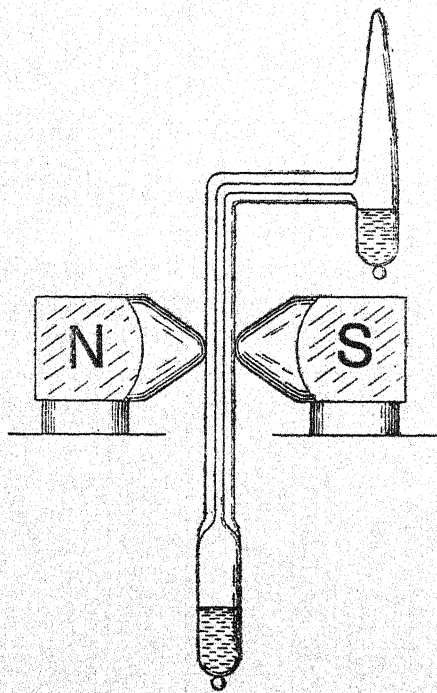
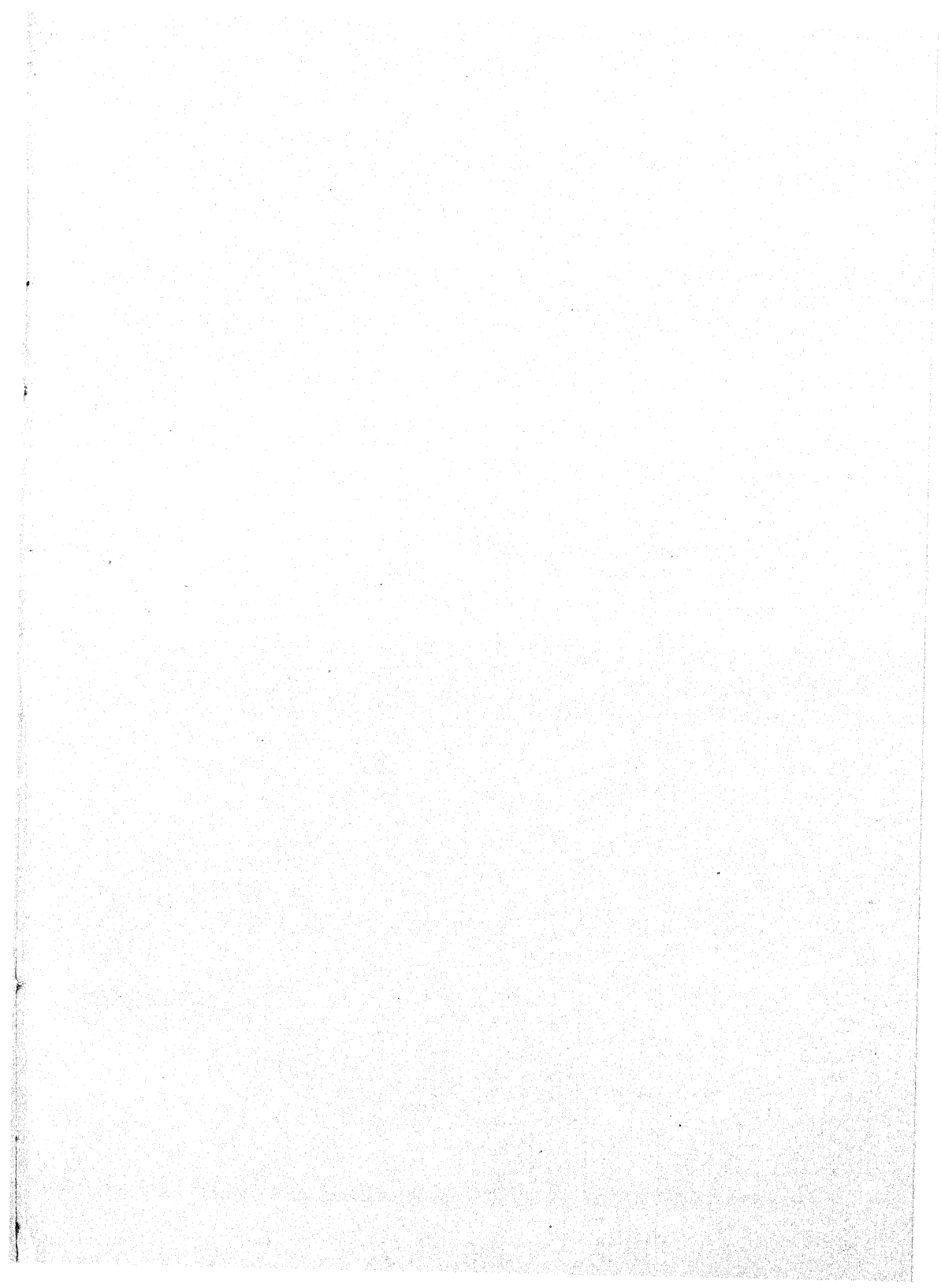


FIG. 1.

that this dissociation of the glass is purely a side issue and a natural consequence when powerful fields and discharges are used. In practice by the employment of such powerful fields and discharges the real effect gets masked and complicated to a considerable degree. To examine this effect of the magnetic field on the radiation, the disruptive action should be reduced to a minimum by the employment of a moderate field and current and even then the tube must be of a material like quartz not liable to suffer such easy decomposition arising from any slight local heating.



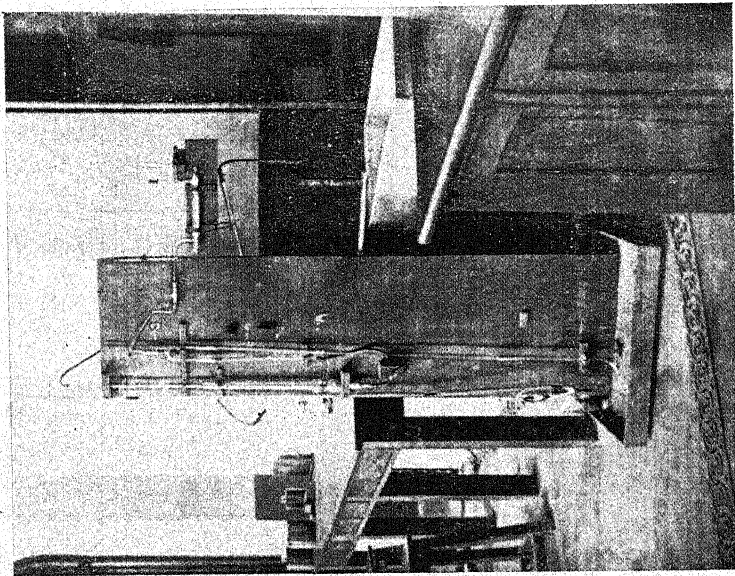


Fig. (1)

An Automatic Mercury Pump

(a)



(b)



(c)



Fig. 2
Effect of Magnetic Field on
General Spectrum,

In my experiments the diameter of the quartz capillary was about 1.5 mm. and the current about 2 milliamperes, the field being of the order of about 5000 C.G.S. units. The arrangement adopted is illustrated in Fig. 1, and the results obtained point to a definite and positive effect of the magnetic field whose nature, however, is not yet quite clear.

In the case of a mercury discharge-tube at a rather low pressure the four lines in the red

$\lambda\lambda$ 6234, 6152, 6123, 6072


appear faintly. But the effect of the magnetic field on these four is different. The line λ 6152 is enhanced very much while the others decrease in brilliancy. In fact at a certain stage the line λ 6152 is practically invisible and is then brought out very brilliantly by the magnetic field. Such an anomalous behaviour in a set of lines belonging to the same element can be attributed only to a positive effect of the magnetic field. It may also be mentioned here that this line λ 6152 of mercury has already appeared abnormal, in that it is a line brought out when helium is present in the tube. This line seems to have some peculiar significance in the spectrum of mercury and deserves a special study. Fig. 2 (a) in the Plate XIII shows the effect in the case of the mercury spectrum and this abnormal line is marked by a dot.

A line of argument to explain this effect led to experiments with the rare gases helium, neon, etc., and their mixtures with the diatomic gases oxygen, hydrogen, etc. The effect of the magnetic field was to enhance very considerably the lines of these monatomic gases, even when they are present in a pure state. When mixed with other diatomic gases the remarkable point is that the lines of those monatomic gases alone are considerably enhanced, while those of the diatomic and others are scarcely affected. Thus in a tube of neon and hydrogen the principal hydrogen lines are not affected at all, while the neon lines are very much enhanced [Fig. 2 (b) in the Plate]. Similarly when a trace of helium is mixed with hydrogen or oxygen and the helium lines are ordinarily not visible at all, the magnetic field brings out the helium lines prominently, leaving the others scarcely affected [Fig. 2 (c)]. These results led to the natural inference by analogy that such enhanced lines are the radiations of an atom while the others may be attributed to the molecule.

The reason why the atomic radiation should be enhanced is not yet known. Thus in the case of mercury the enhanced lines $\lambda 6152$ and others are evidently to be classified as atomic radiations and the unaffected lines are due to the molecules. In such a view it is not meant that these are absolutely pure radiations. It may be that every line is made up of the radiations of both the atom and the molecule, but that the proportions are different in different cases, so that we may explain the highly enhanced radiations as those in which the atomic radiation predominates. The enhancement of the argon lines noted by Kent and Frye is also in keeping with this explanation.

This line of argument led to experiments with sulphur and iodine in a vacuum tube and as is to be expected from the complex of molecules and atoms in all states of aggregation which we know to exist there, the spectrum, under the influence of the magnetic field, showed a variety of marked changes. These cases are under more detailed study at present. Even in the case of the diatomic gases the effects are not simple, but considerably complicated, since their atomic or molecular state and the proportionality of their radiations can be expected to be dependent on pressure and temperature. Even in the case of atmospheric air at low pressure some marked changes are observed identical with quartz or glass tubes, so that the question of the disintegration products of the glass and its complications do not arise at all. Further examination of the phenomenon in greater detail is in progress.

*Cavendish Laboratory,
Cambridge, England;
May 6th, 1920.*



XVI. An Improved Type of Automatic Mercury Pump.

By H. P. Waran, M.A., Government Scholar of the
University of Madras.

(Plate XIII, Fig. 1 and 1(a).)

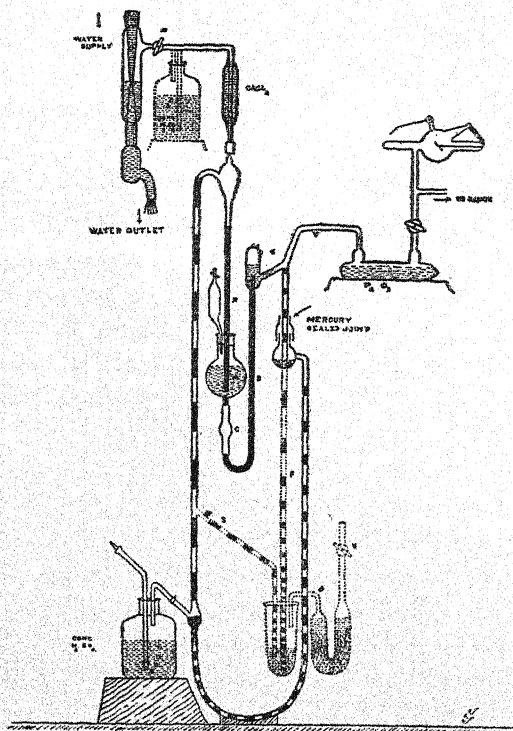
The pump under consideration is of a German design and worked by the mercury falling in drops down a long fall tube catching small pellets of air between as per original Sprengel principle. The pump is automatic in action since the mercury that fell down to the bottom reservoir is transferred back again to the upper reservoir and made to flow again with the help of a filter pump worked off the main water-supply.

A quick acting pump of the type was urgently required in connection with some cathodic deposition experiments and when the pump was constructed as per original design it was found not to work at all. Further experimentation with it showed the necessity for a good deal of improvement to be made on the original design before it could be made working efficiently.

Described in brief the original design of the pump was as follows:—

A wide-necked flask-like vessel *A* with a tubulure at the bottom is the main mercury reservoir and it contains about 2lb. of mercury. On the rubber connection from the tubulure downwards to the bend of the air trap tube *B* is placed a clamp *C* to regulate the flow of mercury. The mouth of *A* is closed with a rubber stopper through which passes the central mercury return tube *R* and a drying tube of calcium chloride. From the bottom of the air trap *I* leads up a tube which branches upwards to the receiver connections and downwards as the main fall tube *F*. This fall tube is about 4 to 4½ ft. in length and terminates in a lower

reservoir, a big stout test tube closed by a three-hole rubber cork through which a side suction tube *S* as well as an air inlet drying tube *D* also passes. The suction tube goes up along the side and connects to the top of the mercury return tube *R*. From the top bulb of this return tube *R*, an exhaust tube with cock *X* is led through a drying bottle *P* of concentrated sulphuric acid, to the suction pipe of a filter pump. The tube *Y* to the receiver goes along



[The dotted lines indicate portions of the original design which do not appear in the finally modified form of the instrument.]

a phosphorus pentoxide tube and terminates at the receiver (a discharge tube) with a side connection to the vacuum gauge.

Now when the cock *W* is closed, the filter pump started and the cock *X* opened, air is rapidly sucked away from the whole system until it reaches the limit of suction of the pump, i.e. about 2 centimetres of mercury pressure. Then the clamp *C* is gradually opened and mercury run into from the trap at *T* and fill up the

bottom test tube reservoir up to the mouth of *S*. Then the rate of fall of mercury is reduced with *C* and cock *W* is opened when a vigorous suction of air is maintained up *S* and all excess mercury coming down *F* in pellets is sucked up along with alternate pellets of air into the tap of *Q* wherefrom it reaches the main reservoir *A* vertically below. Thus the mercury is maintained in automatic circulation and the pellets of mercury going down the fall tube do the exhaustion. This was the original design of the pump.

When such a pump was made and tried it was not found to work quite efficiently and at first the mercury could not be got into automatic circulation at all. Immediately experiments to set it going were begun after temporarily disconnecting the air trap and other accessories.

Instead of a single filter pump a pair of filter pumps were connected in parallel, and even though the upward draught was much more powerful the mercury could not be got into automatic circulation. Further thought on the subject suggested that the main essential point was the necessity to establish a very strong air suction through the side tube to enable the mercury to be sucked up as rapidly as it comes down the central main fall tube. To secure this the vacuum created by the filter pump should be high and speedy and the main drawback of the original design was detected to be the large capacity introduced in the air suction circuit. The drying bottle of concentrated sulphuric acid as well as the bottom reservoir were too big, and once the bottom of the tube *S* got uncovered and the full atmospheric pressure of air rushed in, it took some time for the pump to establish the original vacuum and a powerful draught to take up the mercury from the lower reservoir. It naturally follows that the smaller this capacity, the more quickly the vacuum created and the more powerful the draught. So a small drying tube of calcium chloride is put in between the pump and the suction tube, while the main drying bottle of concentrated sulphuric acid is put in lower down to enable the air to get dried before being sucked into the system. The next improvement was the abolition of the lower reservoir altogether which was an improvement in not only that it decreased the lower capacity still further, but eliminated the difficulty of a 3-holed rubber cork with air leaks and so on at this reservoir. Instead of this reservoir the lower end of the fall tube was curved and made to join up the suction tube

with a small bulb at the beginning of the suction tube with a tap sealed on to it. This tap leads to the drying sulphuric acid bottle and the suction tube of this bottle has another tap sealed on to it partly for regulation and partly to close up the acid from external moisture when the pump is left idle. This improvement prevented the accumulation of mercury at the bottom owing to any maladjustment of the rate of fall and suction, since the moment the mercury level rose up in the bulb to close the air inlet, the excess was sucked up in the powerful draught and then returned to the main reservoir through the tube *R*. With this modification the automatic mercury circulation worked up quite quickly and very vigorously even when only one ordinary locally made glass filter pump was used. The air getting mixed up with the mercury is quite dry and no fouling of the vacuum from vapour pressure is to be expected. It is very useful to observe the general precaution that the bore of the suction tube *S* as well as that of the fall tube should be about a millimetre and not more, to prevent the air escaping by the side of the mercury pellets. These tubes are conveniently made of thick walled capillary glass tubing of internal diameter about a millimetre.


The vacuum air trap is as usual and is an essential adjunct when the highest vacuum is to be used. With these modifications introduced the pump works quite easily and rapidly, perfectly automatic in action, once the rate of fall of mercury is adjusted and the two taps at the bottom opened after preliminary vacuum of about 2 centimetres of mercury has been directly obtained by the filter pump. Quite a cathode ray vacuum is produced in about 30 or 40 minutes everything going well. But working the pump longer, the vacuum was found not to improve further, and it was difficult to push the discharge tube to quite a hard condition. An examination in detail of what is happening in the fall tube at this stage shows the reason why. The fall tube must be scrupulously clean and when the metallic clicks of the falling mercury pellets are noticed it will be found that a good length of air in the fall tube is compressed by the falling column of mercury to quite a tiny disc of air of imperceptible size. This air somehow does not seem to escape out of the tube along with the mercury, but somehow escapes back into the high vacuum system, probably escaping gradually by the side of the falling pellets. This difficulty can be

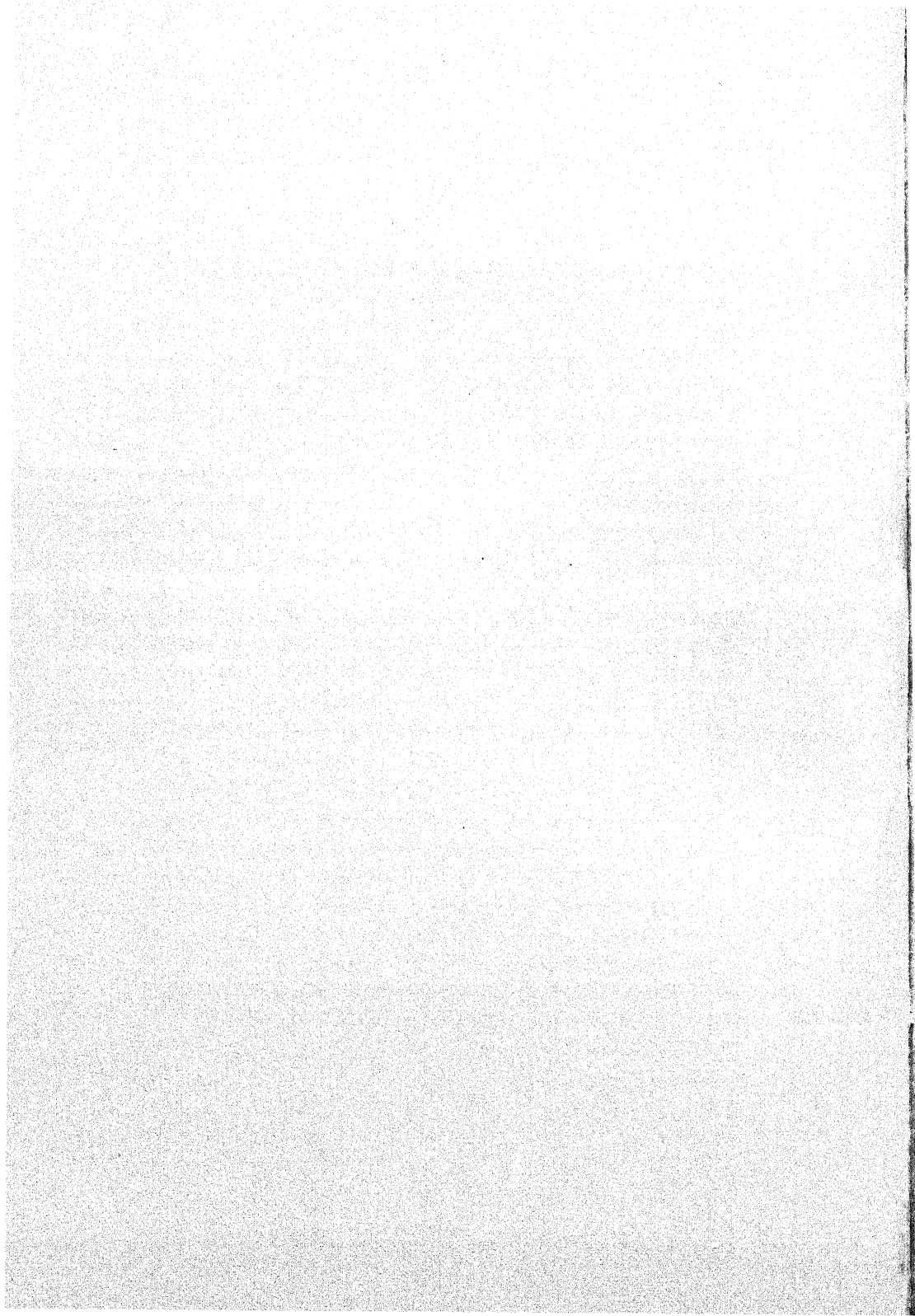
got over if we can prevent the air being compressed into this thin disc-like form.

A suggestion to get over this difficulty is to do this latter high exhaustion in two stages so that in the first stage the air is not compressed but simply pushed into another highly exhausted space to produce the extreme vacuum. Following this idea the fall was made in two stages, the first of about 30 centimetres and the other of about 90 centimetres, and the tip (lower) of the first tube was just touching the mercury level on the reservoir on top of the lower tube as illustrated. In such a system, at the initial stages when the pressure is high, the first fall is not very effective and the real exhaustion takes place by the lower fall tube as usual. But later as the vacuum gets very high and the mercury begins to hammer in the lower fall tube, the rate of flow of mercury is slightly increased to secure two or three pellets going down the first tube at the same instant. This succession of pellets down the first fall are very effective and they push the remaining air from the receiver into the upper reservoir from which they can never escape into the high vacuum so long as the downward stream of mercury continues and is maintained permanently by the filter pump. The little air that collects in the upper reservoir is of course removed whenever it is an appreciable quantity by the lower fall tube. This is the most important of all the improvements and its effect is quite remarkable. With its help the pump exhausts to limits which it could not previously exhaust and in the case of one small locally made X-ray tube with electrodes about 2 inches apart, the discharge could be made to prefer an alternate air gap of about 4 or 5 inches with a six-inch coil.

The above piece of work was done at the Presidency College, Madras, and Fig. 1 in Plate XIII shows the pump during its experimental stages when just got working automatically. Fig. 1(a) is a photograph of the double stage exhaustion attachment for high vacuum work.

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Cambridge, England.*





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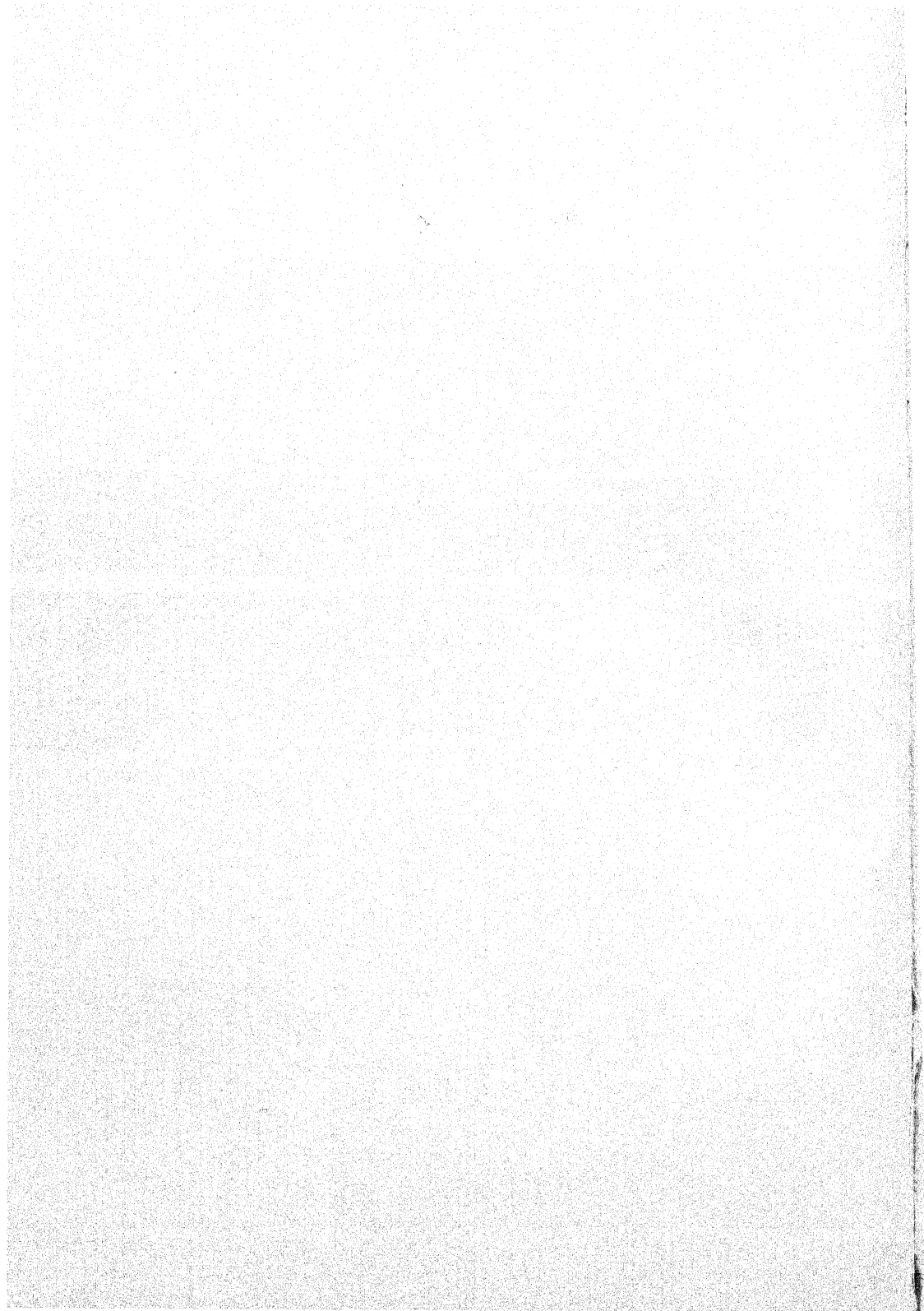
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I. On the Colour and Polarization of the Light scattered by Sulphur Suspensions.

By Bidhubhusan Ray, M.Sc., Lecturer in Optics, Calcutta University.

I. HISTORICAL INTRODUCTION.

The problem of the scattering of light by small suspended particles has attracted a great deal of attention in view of its bearing on the explanation of the blue colour of the sky and of the sea. Some of the earliest experiments on the subject were those made by Tyndall who attempted to explain the polarization and colour of the sky, as due to the scattering of light by very small particles in suspension in the higher atmosphere. He imitated the blue light of the sky in his laboratory by having a cloud of very small particles in a tube, containing Iodide of allyl. The fine particles were formed by the action of light on the liquid.

The late Lord Rayleigh¹ next took up the problem and studied the scattering of light by suspensions of sulphur obtained by adding a drop or two of very dilute sulphuric acid to a weak solution of sodium hyposulphite. At first the scattered light seen in a perpendicular direction is completely polarized, but as the particles began to grow in size, the light is not extinguished in any position of the Nicol but in the position of minimum illumination shows Tyndall's phenomena of the "Residual Blue," proving that for the shortest waves, the polarization of the scattered light is relatively more imperfect. The direction of maximum polarization now becomes oblique and moves back towards the source. Prof. Love² next treated the problem mathematically limiting himself

¹ Scientific Papers, Vol. I and Vol. IV.

² Lond. Math. Soc. Proc. Vol. 30, p. 308.

to the case of spherical particles but without any limitations as to their refractive indices and (so far as the general expressions are concerned) to their radii as well. Owing to the omission of some factors in his equations, some discrepancies arose. The late Lord Rayleigh corrected the results and put them in a very simple form. These equations will be referred to again later on.

Bromwich¹ has given a general solution of the problem of the scattering when a plane simple harmonic electromagnetic wave strikes a sphere. In certain cases a marked simplification is introduced by using spherical polar co-ordinates. When certain assumptions are imposed upon the particles as regards their size and nature, the equations obtained by Bromwich are identical with those obtained by the late Lord Rayleigh, though expressed in a different form. In the case of large perfectly conducting spheres, the formulæ given by Bromwich have been utilised by Messrs. Proudman, Doodson and Kennedy² who have made numerical calculations for the case of $ka=9$ and 10, where $k=\frac{2\pi}{\lambda}$ and a =radius of the spherical particles. Here they found that the curve showing the variation of the intensity of the light scattered in different directions is of an oscillatory character.

In course of a study of the phenomena of the "Residual Blue" in the light scattered by sulphur suspensions obtained by Rayleigh's method, the author was induced to continue his observations to a stage at which the particles had grown much larger in size, and found that oscillations of the intensity of the light scattered in different directions were readily observable. They were somewhat similar to those shown in the curves published in the paper by Messrs. Proudman, Doodson and Kennedy, but the characters of the curves for the light polarized in and right angles to the plane of scattering in this case were found to be interchanged. The difference is not surprising, as in the present instance we are dealing with sulphur particles which are not at all perfectly conducting spheres. It is proposed first to give a general description of the author's observations and then to discuss the results on the basis of the rigorous electromagnetic theory of diffraction.

¹ Phil. Trans. Series A, Vol. 220, 1920.

² Phil. Trans. Series A, Vol. 217, 1917.

2. EXPERIMENTAL RESULTS.

The experimental arrangements required are of the simplest kind. Light from an 1,000 c.p. lamp is focussed by a lens upon a rectangular slit behind which was placed a beaker of clear water; 15 to 20 drops of a very weak solution of sodium thiosulphate is added to the water in the beaker and 4 to 5 drops of very dilute sulphuric acid are then put in and the liquid stirred. A fine precipitate of sulphur slowly forms which scatters the light passing through the liquid. Care should be taken not to add more acid than absolutely necessary as the formation of sulphur particles is then unduly hastened and the phenomena now to be described cannot be satisfactorily observed. The best results are obtained when the quantity of acid is just sufficient to cause the first blue colour to appear after 15 minutes.

As the sulphur suspension gradually forms, the colour and polarization of the light scattered by it change in a remarkable manner. For qualitative observations it is convenient to view the scattered light with the eye placed above the beaker, the illuminated portion of the liquid being as near the surface as possible. For purpose of measurement of the angles of scattering, the scattered light from the centre of the beaker may be observed through the sides of the vessel. The scattered light varies in intensity and colour in different directions and when observed through a Nicol, changes as the Nicol is rotated about its axis.

Lord Rayleigh observed that "In the early stages of precipitation, polarization is complete in a perpendicular direction and incomplete in other directions. After an interval the polarization begins to be incomplete in a perpendicular direction, the light which reaches the eye when the Nicol is in the position of minimum transmission being of beautiful blue colour, much richer than any thing that can be shown in the earlier stages." At this stage the polarisation is more complete in a direction with $\theta =$ positive, i.e. back towards the source. When the particles increase in size, a neutral point comes in near $\theta = 0$ with still further increase, this neutral point moves away towards $\theta = 180^\circ$, and between the neutral direction and $\theta = 0$, the relative intensity of the two components is interchanged, in other words, the state of polarization of the scattered light is now reversed. Lord Rayleigh observed such reversal with blue light only. The author has con-

firmed Lord Rayleigh's work and also extended his investigations to much larger particles. It is found that with larger particles this reversal can be seen with red light and it is possible to follow up the movement of the neutral direction to a stage in which it is nearly $\theta=180^\circ$. Further with white light, the scattered light shows a succession of colours in different directions, these colours being different for the two components of polarization. The observations are tabulated below:—

TABLE I.
Colours of the Scattered Light.

Time from Commencement.	Orientation of the Nicol.	Description of Colour and direction of Scattered Light.	REMARKS.
15 mm.		*Blue colour (Residual blue) max. being at 90°	Light scattered nearly parl. to the direction of transmission is white but slightly tinged blue.
	⊥	White patch (but slightly bluish)	
30 min.		Brownish red : *blue $\theta=0$ to $\theta=60^\circ$: $\theta=60^\circ$ to $\theta=135^\circ$	Transmitted light is yellowish white.
	⊥	slightly brownish tint $\theta=0$ to $\theta=85^\circ$	
45 min.		Brownish red colour disappears and greenish blue appears extending from $\theta=0$ to $\theta=120^\circ$	" "
	⊥	Blue : Brownish red $\theta=0$ to $\theta=85^\circ$: $\theta=85^\circ$ to $\theta=135^\circ$	
1 hr.		Greenish yellow from $\theta=0$ to $\theta=110^\circ$ other colours cannot be distinguished.	" "
	⊥	Brownish red : *green $\theta=0$ to $\theta=70^\circ$: $\theta=70^\circ$ to $\theta=112^\circ$ Red $\theta=112^\circ$ to 140°	

|| component is the component having its vibrations in the plane containing the direction of the incident ray and the direction of observation. The vibration is perpendicular to the direction of observation.

⊥ is the component having its vibrations perpendicular to the above plane. The vibration is also perpendicular to the direction of observation.

* Indicates that the colour is very pronounced.

TABLE II.
Colours of the Scattered Light.

Time from Commencement.	Orientation of the Nicol.	Description of Colour and direction of Scattered Light.
1 hr. 12 min.		Brown : Green : The transmitted light $\theta=0$ to $\theta=65^\circ$: $\theta=65^\circ$ to $\theta=110^\circ$: turns a bit reddish.
	\perp	Bluish green: Crimson red: *Greenish yellow $\theta=0$ to $\theta=60^\circ$: $\theta=60^\circ$ to 110° : $\theta=100^\circ$ to 130° Rosy red, : the colour being pro- $\theta=130^\circ$ to 180° : nounced at $\theta=160^\circ$
		Yellowish green: Light brown: Greenish white: $\theta=0$ to $\theta=65^\circ$: $\theta=65^\circ$ to 97° : $\theta=97^\circ$ to 120° : Transmitted light is reddish.
1 hr. 22 min.	\perp	Light red : *Greenish yellow: *Orange red : Yellow $\theta=0^\circ$ to 60° : $\theta=60^\circ$ to 85° : $\theta=85^\circ$ to 114° : $\theta=114^\circ$ to 140° Red light till $\theta=180^\circ$, colour most pronounced at 160°
		Greyish green: Brown red : Green : Trans- $\theta=0$ to $\theta=88^\circ$: $\theta=88^\circ$ to 117° : $\theta=117^\circ$ to 140° : mitted light s red
1 hr. 35 min.	\perp	Light red : Yellow : Orange red : Yellow $\theta=0$ to $\theta=55^\circ$: $\theta=55^\circ$ to 92° : $\theta=92$ to 117° : $\theta=117^\circ$ to 135° Transmitted light is red, colour most pronounced at $\theta=165^\circ$.

|| component is the component having its vibrations in the plane containing the direction of the incident ray and the direction of observation. The vibration is perpendicular to the direction of observation.

\perp is the component having its vibrations perpendicular to the above plane. The vibration is also perpendicular to the direction of observation.

* Indicates that the colour is very pronounced.

A few observations were made of the direction of the maxima in the scattered light in this case using red and yellow glasses respectively to approximately monochromatize light. The following results are tabulated for the case in which the Nicol is in the \perp position. The observations clearly indicate that the colours of the scattered light arise mainly from the fact that the directions of the maxima and the minima are different for different wave lengths.

Angle θ = Position of the Maximum.

Red	45°	105°	165°
Yellow		75°	125°

3. THEORETICAL.

Let us suppose that a beam of unpolarized light falls on a spherical obstacle with its centre as the origin and also let the light travel in the negative direction along the axis of Z . Suppose we confine our observation to the horizontal plane (i.e. plane containing Z, X) at a distance r from the centre and making an angle θ with incident beam. We can observe both the horizontal and the vertical components of the scattered light with the help of a Nicol at the point in question.

If X, Y and Z denote the electric forces parallel to the three axes in the scattered wave then the vertical component of the scattered light is denoted by Y while the horizontal one by

$$\frac{xZ - zX}{r}$$

Love's solution as corrected and modified by the late Lord Rayleigh gives the following expressions for the two components.

$$Y = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n+1}{n(n+1)} \left[M_n \{ \mu P_n' - n(n+1) P_n \} + N_n P_n' \right] \frac{e^{ik(ct-r)}}{kr}$$

$$\frac{xZ - zX}{r} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n+1}{n(n+1)} \left[N_n \{ \mu P_n' - n(n+1) P_n \} + M_n P_n' \right] \frac{e^{ik(ct-r)}}{kr}$$

In these equations, $k = \frac{2\pi}{\lambda}$, λ being the wave length of the incident light, $\mu = \cos \theta$ and P_n or $P_n(\mu)$ is a zonal harmonic of degree n , whose axis is the axis of Z . Mod Y and Mod $\frac{xZ - zX}{r}$ give the amplitudes of the two components and their squares give the intensities. M_n and N_n are functions of the size and optical proportions of the spherical particles. The complete expression for

N_n is

$$\frac{K\psi_{n-1}(\eta) - \left\{ (K-1) \frac{n}{2n+1} + \frac{\psi_{n-1}(\eta^1)}{\psi_n(\eta^1)} \right\} \psi_n(\eta)}{K - E_{n-1}(\eta) + \left\{ (K-1) \frac{n}{2n+1} + \frac{\psi_{n-1}(\eta^1)}{\psi_n(\eta^1)} \right\} E_n(\eta)}$$

and

M_n

$$\frac{\psi_{n-1}(\eta) - \frac{\psi_{n-1}(\eta^1)}{\psi_n(\eta^1)} \psi_n(\eta)}{-E_{n-1}(\eta) + \frac{\psi_{n-1}(\eta^1)}{\psi_n(\eta^1)} E_n(\eta)}$$

The expression for M_n is obtained by substituting μ the magnetic permeability, instead of K . In optical problems we may take $\mu=1$ so the expression for M_n stands as above.

K is the dielectric constant of the material composing the spherical particle, that of the surrounding medium being supposed equal to unity. K may be substituted for m^2 where m is the refractive index of the material composing the sphere, relatively to the surrounding medium, and $\eta^1 = m \cdot \eta$.

$$\psi_n = (-1)^n 1.3.5.7.9 \dots (2n+1) \left(\frac{1}{\eta} \frac{d}{d\eta} \right)^n \frac{\sin \eta}{\eta}$$

and

$$\bar{E}_n = (-1)^n 1.3.5.7.9 \dots (2n+1) \left(\frac{1}{\eta} \frac{d}{d\eta} \right)^n \frac{e^{-i\eta}}{\eta}$$

so that

$$E_n(\eta) = \Psi_n(\eta) - i\psi_n(\eta)$$

when real and imaginary parts are separated.

4. CALCULATIONS WITH SULPHUR PARTICLES.

The late Lord Rayleigh in the same paper calculated the values of $\text{Mod } Y$ and $\text{Mod } \frac{xZ - zX}{r}$ for various values of η , not greater than 2.25 assuming $K=2.25$ (an arbitrary value). He also suggested that "It would be possible to follow these calculations to greater values of η , and such an extension would not be without interest, but the arithmetical work would soon become heavy. Also, without increasing η , the refractive index might be varied." It was thought that in the case of sulphur the equations given above would explain the oscillations of the colours of the scattered light in different directions if numerical calculations were made with the appropriate data. As these colour fluctuations were observed when the particles were larger than those responsible for the Tyndall's "Residual blue" phenomenon, it was only reasonable that calculations should be made for larger values of η or (ka) where

a =radius of the particle. Refractive index for sulphur particles is 1.95 and that of water 1.33. K was accordingly taken to $\left(\frac{1.95}{1.33}\right)^2$ and the numerical values were computed on the assumption that $Ka=5$.

In order to calculate the values of $E_n(\eta)$, the sequence formula was used

$$E_{n+1} = \frac{(2n+1)(2n+3)}{\eta^3} [E_n - E_{n-1}]$$

starting from E_0 and E_1 . This method is satisfactory as regards the real part of $E_n(\eta)$, but as the imaginary part tends to equality, any error, then may creep in, is multiplied at the next step by the large factor $(2n+1)(2n+3)$. The difficulty can be overcome in the following manner when the convergence is good. We may calculate the value of ψ_9 and ψ_{10} by a straight forward method very accurately. Having obtained them we may then use the sequence formula in a reverse direction to find the lower values of ψ , without any loss of accuracy.

The values of Ψ and ψ were calculated for $\eta=5$ and tabulated below.

Tables of Ψ and ψ .

	Ψ	ψ
0	.056748	— .19175
1	— .108264	— .05704
2	— .099009	.080827
3	.01295	.19302
4	.28214	.28273
5	1.066	.35524
6	4.484	.41478
7	26.664	.46443
8	226.236	.506412
9	2578.44	.5424

Tables for M_n and N_n .

	M_n	N_n
1	— .4978— $i \times .4529$	— .5555— $i \times .7061$
2	— .1713— $i \times .0030$	— .2881— $i \times .0913$
3	— .2423— $i \times .0626$	— .3926— $i \times .8098$

	M_n	N_n
4	$-.3453 - i \times .8616$	$.0287 - i \times .9992$
5	$.4998 - i \times .5131$	$.4812 - i \times .3642$
6	$.0554 - i \times .0031$	$.1476 - i \times .0223$
7	$.00475 - i \times .00002$	$.0124 - i \times .0015$
8	$.00036 - i \times 0$	$.0013 - i \times 0$
9	$.00002 - i \times 0$	$.0001 - i \times 0$

The quantities,

$$\left\{ \frac{2n+1}{n(n+1)} \mu P'_n - (2n+1)P_n \right\} \text{ or } A_n \text{ and } \frac{2n+1}{n(n+1)} P'_n \text{ or } B_n$$

are functions of n and η . Their logarithmic values for various values of μ have been tabulated by Rayleigh¹ up till $n=6$ and their actual numerical values have been given by Paris² for $n=5$. Calculations have been pushed up to $n=9$ yet the tables are not given as the values A_n, M_n, B_n, N_n , etc., decrease rapidly.

Table for Mod Y and Mod $\frac{xZ-zX}{r}$

μ	$\left(\text{Mod } \frac{xZ-zX}{r} \right)^2$	$(\text{Mod } Y)^2$
1	11.88	11.88
.95	8.12	2.15
.75	.79	6.76
.6	2.1	9.12
.5	3.54	11.91
.4	8.23	14.26
.25	13.71	4.52
0	5.46	14.01
-.25	9.15	21.25
-.4	14.8	11.32
-.5	19.01	5.54
-.75	42.18	17.44
-.95	110.72	127.79
-1	376.57	376.57

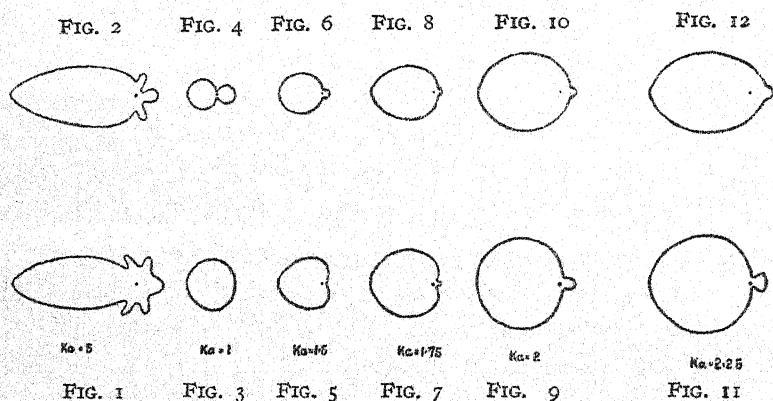
Polar diagrams have been drawn (Figs. 1 and 2) representing (intensity)¹ or Mod Y and Mod $\frac{xZ-zX}{r}$. The form of the graphs shows the remarkable resemblance between the theoretical and

¹ Royal Soc. Proc. Ser. A, Vol. 84

² Phil. Mag. Vol. 30, 1915.

observed phenomena. It shows that the \perp component of the scattered light shows 3 oscillations while the \parallel one shows only 2. It also shows that the max. and min. of Fig. 1 (\perp Component) are sharper and steeper than those of Fig. 2 (\parallel Component). This also is the actual case, where the number of oscillations is greater in the \perp component than the \parallel one; the colours in the former component are more prominent.

The graphs drawn in Figs. 1 and 2 evidently correspond more or less closely to the stage after 1 hr. and 35 minutes in Table II.



Also from the graph in Figs. 1 and 2 it is seen that the maxima are more pronounced when θ approaches 180° . This is in agreement with observation as the intensity of the colour increases in that direction. In order to get a clear idea of how the light scattering alters with the size of the particles for the two states of polarization, the numerical data given by the late Lord Rayleigh¹ for $ka=1$, $ka=1.5$, $ka=1.75$, $ka=2$ and $ka=2.25$ have been plotted as polar diagrams in Figs. 3 to 12. It will be noticed that the \parallel component which has no intensity when $\theta=90^\circ$ and ka is small shows fewer and less marked fluctuations with increasing values of ka . The tendency is clear even when $ka=1.75$ and becomes patent for still larger values of ka .

The radius of the particles in Table II and after 1 hr. 30 min. from commencement was found to range between 0.3μ to 0.5μ being thus of the same order of magnitude as that indicated by the character of the light scattered by them.

¹ Proc. Roy. Soc. Ser. A, Vol. 84, 1911.

5. SUMMARY AND CONCLUSION.

1. The colour and polarization of the light scattered in different directions by very fine suspensions of sulphur have been studied by the late Lord Rayleigh. In the present paper, the work is extended to much larger particles and some very remarkable results have been obtained. It is found that for the larger particles the curves representing the intensity of the light scattered in different directions become of a oscillatory character, the phenomena being markedly different for light having its electric vector \parallel and \perp to the plane of scattering containing the incident and scattered rays.

2. Rayleigh observed that at the early stages, as the particles increase in size the position of maximum polarization shifts towards the source. With still further increase in the size of the particles, a neutral direction appears nearly at $\theta=0$ which gradually begins to shift towards $\theta=180^\circ$. Between the neutral direction and $\theta=0^\circ$, the state of the polarization of the scattered light is reversed. Rayleigh observed such reversion only with blue light. The author has pushed the investigation to larger particles and observed such reversal with red light and traced further the movement of the position of the neutral direction to about $\theta=180^\circ$.

3. In the earlier stages, as the particles form, it is found that the \parallel component shows more pronounced fluctuations of intensity and colour, than the \perp component, the latter showing hardly any trace of variation of intensity with direction. As the particles increase in size, the position is reversed (see Tables I and II of the paper). The \perp component now shows a larger number of oscillations, and the colours in it are more prominent than those of the \parallel component. The colour sequence alters in the manners indicated in Tables I and II. The intensity curves are somewhat like those published by Messrs. Proudman, Doodson and Kennedy¹ for perfectly conducting particles except that the characters of the curves for \parallel and \perp component are here interchanged. The difference is due to the fact that in this case we are dealing with particles of sulphur which are transparent, and not with perfectly reflecting particles.

¹ Phil. Trans. Vol. 217, 1917.

4. In order to explain the phenomena, the theoretical formulæ have been numerically computed for the case in which $ka=5$ and the results are found to agree with the observed facts.


5. The colours of the transmitted light as seen by Porte and Keen¹ will be theoretically discussed in a separate paper.

The investigation here described was carried out in the Palit Laboratory at the University College of Science and the author is indebted to Prof. C. V. Raman, Palit Professor of Physics, for the unfailing interest he has taken during the progress of the work.

*University College of Science,
92, Upper Circular Road,
Calcutta.*

The 5th May, 1921.

¹ Roy. Soc. Proc. Vol. 89, Ser. A.



II. On the Vibrations of the Pianoforte-String.

By Panchanon Das, M.Sc., Research Scholar in the Indian Association for the Cultivation of Science.

The main features of the vibrations of the pianoforte-string were first elucidated by Helmholtz, who assumed the force exerted by the hammer during impact to be of the form $F \sin pt$. Next, Kaufmann¹ discussed it on different lines with the aid of functional solutions of the equation of wave-motion, but his treatment in the most important case, viz. that of a string struck not far from one end, was only approximate. Prof. C. V. Raman² and B. Banerji, worked out the problem in a more complete manner by regarding the motion during impact as that of a loaded string and by taking into account the discontinuous changes in the pressure of the hammer due to the successive reflections of the impulse from the ends of the string. Kaufmann's work can also be extended so as to give more accurate results by working out the successive functional solutions. This method is not so cumbersome as might be apprehended; in fact the results are very simple and elegant in form.

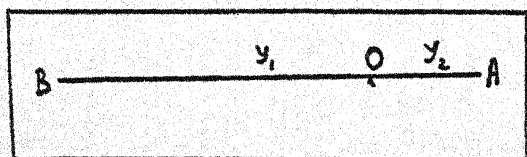


FIG. 1.

Fig. 1, represents a pianoforte-string. O is the position of the point where the hammer strikes the string and A is the fixed end of the same, the length OA being equal to a . We assume that the length a is so small, or rather the other end of the string

¹ Annalender Physik Band 54 (1895). ² Proc. Royal Society A., Vol. 97 (1920).

attached to the bridge B is so remote from O , that the impact ceases before the wave reflected from the bridge-end reaches O . For the present we restrict ourselves to the case of a rigid hammer for the simplicity of its treatment; the case of an elastic hammer will be discussed later on. The following are the symbols used in the paper :—

- m = Mass of the hammer.
 ρ = linear density of the string.
 T = tension of the string.
 c = wave-velocity in the string.
 v_0 = initial velocity of the hammer.
 t = variable time.

We take the x -axis along OA , and for simplicity suppose the vibrations to be in a plane passing through OA and \perp to OA . Let the displacement at any point of the string at any instant t be y_2 for positive values of x , y_0 at O and y_1 for negative values of x .

We divide the duration of the impact into a series of epochs determined by $0 < ct < 2a$, $2a < ct < 4a$, $4a < ct < 6a$, and so on, so that each of these epochs corresponds to the time taken by a pulse from O in returning to O after reflection from the end A . It will be found, as easily follows from general considerations, that the motion of the string is different at different epochs. So it is necessary to investigate the motion in each of these succeeding epochs.

I. Epoch ($0 < ct < 2a$):—

The equation of motion of the string on each side of O is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2},$$

and that of the hammer is

$$m \frac{d^2 y_0}{dt^2} = T \left(-\frac{\partial y_1}{\partial x} + \frac{\partial y_2}{\partial x} \right)_{x=0} \dots\dots\dots (1)$$

As the hammer impinges on the string, two waves of the same form start in both directions of the string, so that the displacements are

$$\left. \begin{aligned} y_1 &= f_1(ct+x) \\ y_2 &= f_1(ct-x) \\ y_0 &= f_1(ct) \end{aligned} \right\} \dots\dots\dots (2)$$

However from after $ct=a$, a pulse corresponding to the reflection of $f_1(ct-x)$ from the end A will affect the value of y_2 , but that pulse will not reach O , until $ct=2a$, so that if we confine our attention to the immediate neighbourhood of O , the value assumed in (2) is sufficiently representative in the First Epoch.

Substituting (2) in (1) we get,

$$mc^2 f_1''(ct) + 2T f_1'(ct) = 0, \\ \text{or} \quad f_1''(ct) + k f_1'(ct) = 0, \dots\dots\dots (3)$$

where
$$k = \frac{2T}{mc^2} = \frac{2\rho}{m}.$$

The first integral is

$$f_1'(ct) = C e^{-kct}.$$

When $t=0$, $y_0 = v = c f_1'(c) = cC$.

$$\therefore C = \frac{V_0}{c}.$$

Thus,
$$f_1'(ct) = \frac{V_0}{c} e^{-kct}. \dots\dots\dots (4)$$

II. Epoch ($2a < ct < 4a$) :—

The wave $y_2 = f_1(ct-x)$ of the first epoch is reflected back as a wave $-f_1(ct+x-2a)$, and reaches O at $ct=2a$. This is easily seen from the fact, that $f_1(ct-x) - f_1(ct+x-2a)$ produces a node at $x=a$. We shall suppose that this reflected wave is transmitted unaltered through O , but excites fresh waves in the positive and negative directions of the x -axis, while the original wave f_1 continues. To satisfy the condition of continuity of displacement at O , these two waves, thus supposed to be excited, must be of the same functional form, i.e. $f_2(ct-x)$ and $f_2(ct+x)$ respectively. We then write

$$\left. \begin{aligned} y_1 &= f_1(ct+x) + f_2(ct+x) - f_1(ct+x-2a) \\ y_2 &= f_1(ct-x) + f_2(ct-x) - f_1(ct+x-2a) \\ y_0 &= f_1(ct) + f_2(ct) - f_1(ct-2a) \end{aligned} \right\} \dots\dots\dots (5)$$

Substituting these in (1) and simplifying the results with the help of (3), we get

$$f_2''(ct) + k f_2'(ct) = f_1''(ct-2a) \dots\dots\dots (6)$$

Writing z for ct , we get the first integral,

$$\begin{aligned} f_2'(z) e^{kz} &= C_2 + \int f_1''(z-2a) e^{kz} dz \\ &= C_2 + f_1'(z-2a) - k \int f_1'(z-2a) e^{kz} dz. \end{aligned}$$

Substituting from (4) we get

$$f_2'(z)e^{kz} = C_2' + \frac{V_0}{c}e^{2ak} - \frac{V_0K}{e}e^{2ak}(z-2a).$$

Or,
$$f_2'(z) = C_2'e^{-kz} + \frac{V_0}{c}e^{-k(z-2a)} \left\{ 1 - k(z-2a) \right\}.$$

At $ct=2a$, the value of y_0 calculated from the I Epoch equals that calculated from the II Epoch.

Thus,
$$f_1'(2a) + f_2'(2a) - f_1'(0) = f_1'(2a)$$

or
$$f_2(2a) = f_1'(0) = \frac{V_0}{c} \dots \dots \dots (7)$$

Hence
$$C_2' = 0, \dots \dots \dots (7a)$$

$\therefore f_2'(ct) = \frac{V_0}{c}e^{-k(ct-2a)} \left\{ 1 - k(ct-2a) \right\} \dots \dots \dots (8)$

III. Epoch ($4a < ct < 6a$):—

We now proceed to apply the same method to the third and subsequent epochs. The pulse $f_2(ct-x)$ is reflected from the end A as $-f_2(ct+x-2a)$ and this after incidence on the hammer gives rise to two new pulses $f_3(ct-x)$ and $f_3(ct+x)$ along OA and OB respectively. The pulses f_1 and f_2 and their reflections still continue. Thus, the displacements are

$$\left. \begin{aligned} y_1 &= f_1(ct+x) + f_2(ct+x) + f_3(ct+x) \\ &\quad - f_1(ct+x-2a) - f_2(ct+x-2a) \\ y_2 &= f_1(ct-x) + f_2(ct-x) + f_3(ct-x) \\ &\quad - f_1(ct+x-2a) - f_2(ct+x-2a) \\ y_0 &= f_1(ct) + f_2(ct) + f_3(ct) \\ &\quad - f_1(ct-2a) - f_2(ct-2a) \end{aligned} \right\} \dots \dots \dots (9)$$

Substituting these in (1) and simplifying the result from (3) and (6) we get

$$f_3''(ct) + kf_3'(ct) = f_2''(ct-2a) \dots \dots \dots (10)$$

The first integral is

$$\begin{aligned} f_3'(z)e^{kz} &= C_3 + e^{kz}f_2'(z-2a) - k \int e^{kz}f_2'(z-2a)dz \\ &= C_3' + \frac{V_0}{c}e^{4ak} \left\{ 1 - k(z-4a) \right\} \\ &\quad - \frac{V_0K}{c}e^{4ak} \left\{ (z-4a) - k \frac{(z-4a)^2}{2} \right\}. \end{aligned}$$

Equating again the values of y_0 from the II and III Epochs at $ct=4a$, we get

$$f_3'(4a) = f_2'(2a) = \frac{V_0}{c}.$$

$$\therefore C_3' = 0 \dots \dots \dots (10a)$$

Hence,

$$f_3'(ct) = \frac{V_0}{c} e^{-k(ct-4a)} \left\{ 1 - 2k(ct-4a) + \frac{k^2}{2}(ct-4a)^2 \right\} \dots \dots \dots (11)$$

IV. Epoch ($6a < ct < 8a$):—

Exactly similar arguments show that, a new pulse $f_4(z)$ is called into play.

The equation of motion reduces to

$$f_4''(z) + kf_4'(z) = f_3''(z-2a) \dots \dots \dots (12)$$

Integrating, we get

$$f_4'(z)e^{kz} = C_4' + \frac{V_0}{c} e^{6ak} \left\{ 1 - 3k(z-6a) + \frac{3k^2}{2}(z-6a)^2 - \frac{k^3}{3}(z-6a)^3 \right\} \dots (13)$$

Again, from the continuity of velocity at $ct=6a$ at 0, we get

$$f_4'(6a) = f_3'(4a) = \frac{V_0}{c} \dots \dots \dots$$

$$\therefore C_4' = 0 \dots \dots \dots (13a)$$

Hence,

$$f_4'(ct) = \frac{V_0}{c} e^{-k(ct-6a)} \left\{ 1 - 3k(ct-6a) + \frac{3k^2}{2}(ct-6a)^2 - \frac{k^3}{3}(ct-6a)^3 \right\} \dots (14)$$

Generalisation:—

It will thus be seen that the end of every epoch, say the n^{th} , a new pulse $f_{n+1}(z)$ is brought into existence, and its form is given by the differential equation

$$f_{n+1}''(z) + kf_{n+1}'(z) = f_n''(z-2a) \dots \dots \dots (15)$$

as is seen from the results (6), (10) and (12).

Also, the suitably adjusted constant of integration vanishes, as is seen from (7a), (10a) and (13a).

Thus comparing the forms of f_2' , f_3' , f_4' we deduce that corresponding to the V Epoch ($8a < ct < 10a$),

$$f_5'(z) = \frac{V_0}{c} e^{-k(z-8a)} \left\{ 1 - 4k(z-8a) + \frac{6k^2}{2}(z-8a)^2 - \frac{4k^3}{3}(z-8a)^3 + \frac{k^4}{4}(z-8a)^4 \right\}.$$

Generalising the result by the method of induction we have, for the $(n+1)^{th}$ epoch, ($n.2a < ct < (n+1).2a$),

$$f_{n+1}(z) = \frac{v_0}{c} e^{k(z-n.2a)} \left\{ 1 - {}^nC_1 \cdot k(z-n.2a) + {}^nC_2 \cdot \frac{k^2}{2} (z-n.2a)^2 - {}^nC_3 \cdot \frac{k^3}{6} (z-n.2a)^3 + \dots + (-1)^n \frac{k^n}{n!} (z-n.2a)^n \right\} \dots \dots \dots (16)$$

The velocity of the hammer in the $(n+1)^{th}$ epoch is given by

$$\frac{\dot{y}_0}{c} = \sum_{r=1}^{r=n+1} f'_r(z) - \sum_{r=1}^{r=n} f'_r(z-2a) \dots \dots \dots (17)$$

where $z=ct$.

The acceleration is given by

$$\begin{aligned} \frac{\ddot{y}_0}{c^2} &= \sum_{r=1}^{r=n+1} f''_r(z) - \sum_{r=1}^{r=n} f''_r(z-2a) \\ &= f''_1(z) + \sum_{r=2}^{r=n+1} \left\{ f''_r(z) - f''_{r-1}(z-2a) \right\}. \end{aligned}$$

From (15) this reduces to

$$\frac{\ddot{y}_0}{c^2} = f''_1(z) - k \sum_{r=2}^{r=n+1} f'_r(z).$$

Hence,

$$\begin{aligned} \ddot{y}_0 &= -kv_0 c \left[e^{-kct} + e^{-k(ct-2a)} \left\{ 1 - k(ct-2a) \right\} \right. \\ &\quad + e^{-k(ct-4a)} \left\{ 1 - 2k(ct-4a) + \frac{k^2}{2} (ct-4a)^2 \right\} \\ &\quad + e^{-k(ct-n.2a)} \left\{ 1 - nk(ct-n.2a) + \frac{n(n-1)}{2} \cdot \frac{k^2}{2} (ct-n.2a)^2 \right. \\ &\quad \left. \left. + (-1)^n \frac{k^n}{n!} (ct-n.2a)^n \right\} \right] \dots \dots \dots (18) \end{aligned}$$

It now remains to interpret these results so as to find expressions for the motion of the string and the pressure of the hammer during impact.

From general notions, it would appear that as the hammer impinges on the string, discontinuous changes of velocity v_0 travel in both directions along the string. The discontinuity travelling towards the fixed end A reaches that point and after reflection there, again reaches O , where it undergoes a second reflection at the hammer and then travels back towards A and so on. This is easily seen from the expression for \dot{y}_2 .

The pressure F of the hammer is $m\ddot{y}_0$. Then, since $k \equiv \frac{2\rho}{m}$, we have from (18)

$$F = -2\rho v_0 c \left[e^{-kct} + e^{-k(ct-2a)} \left\{ 1 - k(ct-2a) \right\} + e^{-k(ct-4a)} \left\{ 1 - 2k(ct-4a) + \frac{k^2}{2}(ct-4a)^2 \right\} \right] \dots \dots (19)$$

Neglecting the sign of F for the time being, the pressures in the successive epochs are:—

$$\begin{aligned} F_I &= 2\rho v_0 c e^{-kct} \\ F_{II} &= 2\rho v_0 c [e^{-kct} + e^{-k(ct-2a)} \{ 1 - k(ct-2a) \}] \\ F_{III} &= 2\rho v_0 c \left[e^{-kct} + e^{-k(ct-2a)} \left\{ 1 - k(ct-2a) \right\} + e^{-k(ct-4a)} \left\{ 1 - 2k(ct-4a) + \frac{k^2}{2}(ct-4a)^2 \right\} \right], \text{ etc.} \end{aligned}$$

It will thus be seen, that at the end of every epoch, the pressure increases by a sudden jump of magnitude $2\rho v_0 c$.

As a typical case, the variation of the pressure of the hammer is worked out and plotted in Fig. 2.

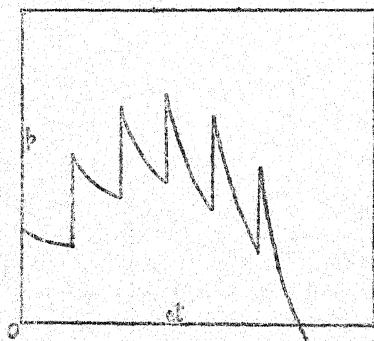


FIG. 2.

It will be seen that these graphs bear a close resemblance to those theoretically obtained by Profs. Raman¹ and Banerji by a different method. The duration of contact is obviously given by the lengths intercepted between the origin and the point of intersection of the curve with the X-axis, divided by C .

¹ *Loc. cit.*

In this investigation, it has been assumed that the impact ceases before the reflection from the bridge-end reaches O . However, when the point O is not very near A , a solution embracing the reflections from both ends can be obtained in much the same manner.

A more detailed investigation of the duration of contact and the amplitudes of the harmonics, as well as the experimental verifications, now in progress, will be given in a later paper.

Lastly, I must add that my best thanks are due to Professor C. V. Raman for his illuminating suggestions and the helpful interest he has taken in this paper.



III. On the Flow of Energy near an Optical Focus.

By T. K. Chinmayanandam, M.A., Director, Government Observatory, Colaba, Bombay.

In continuation of his paper on the Flow of Energy round a perfectly reflecting cylinder,¹ the writer considered it would be interesting to work out the corresponding results for the field in the immediate neighbourhood of an optical focus. For simplicity, the case is taken of a cylindrical wave converging to a linear focus after passing through an aperture of angular width 2σ . The wave is supposed to be plane polarised, the electric intensity lying in the plane of the paper and the magnetic intensity at right angles to that plane.

Let $PY'P$ represent the section of the wave by the plane of the paper on emerging from the aperture, and OX the section of the focal plane. Let the coordinates of a point R in the neighbourhood of the focus be ξ, η with reference to OX, OY as axes of

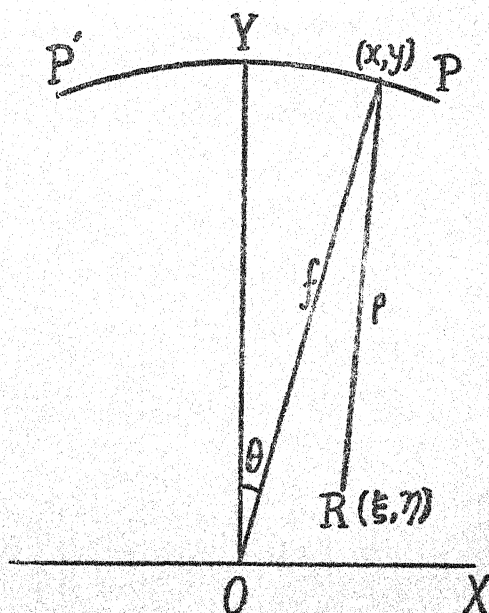


FIG. 1.

¹ Phil. mag. Vol. XXXVII, Jan. 1919 p. 9.

coordinates, and let x, y be the coordinates of an element of the wave-surface $P'YP$. Then if ρ be the distance of the point R from the element (x, y) , the disturbance at R due to the latter may be represented by a periodic expression of the form $\cos k(at - \rho)$ where $k = 2\pi/\lambda$. Expressing ρ in terms of the coordinates chosen.

we have

$$\begin{aligned}\rho^2 &= (x - \xi)^2 + (y - \eta)^2 \\ &= f^2 - 2x\xi - 2y\eta + \xi^2 + \eta^2, \quad f \text{ being the focal length} \\ &= f^2 - 2s\xi - 2f\eta \cos \theta + \xi^2 + \eta^2 \text{ approx.}\end{aligned}$$

where s is the length of the arc measured from Y to the point (x, y) along $P'YP$, and θ the angle which that arc subtends at O . On further transformation

we get

$$\begin{aligned}\rho^2 &= (f - \eta)^2 + \xi^2 - 2s\xi + 2f\eta(1 - \cos \theta) \\ &= \rho_0^2 - 2s\xi + \frac{\eta s^2}{f} \text{ since } s = f\theta = f \sin \theta \text{ approx.}\end{aligned}$$

Hence

$$\rho = \rho_0 - \frac{\xi}{\rho_0} s + \frac{\eta}{2\rho_0^2} s^2. \text{ (approx).}$$

ρ_0 being the distance of R from Y .

Or we may write, substituting f for ρ_0 in the small terms,

$$\rho = \rho_0 - \frac{\xi}{f} s + \frac{\eta}{2f^2} s^2. \text{ (approx).}$$

The total disturbance at R due to the whole wave will be given by

$$\int_{-\sigma}^{\sigma} \cos k \left(at - \rho_0 + \frac{\xi}{f} s - \frac{\eta}{2f^2} s^2 \right) ds \dots \dots \dots (1).$$

This will also represent the resultant magnetic intensity at R .

The electric intensity at the point will have for its components

$$E_x = \int_{-\sigma}^{\sigma} \cos \phi \cos k \left(at - \rho_0 + \frac{\xi}{f} s - \frac{\eta}{2f^2} s^2 \right) ds \dots \dots \dots (2)$$

$$E_y = \int_{-\sigma}^{\sigma} \sin \phi \cos k \left(at - \rho_0 + \frac{\xi}{f} s - \frac{\eta}{2f^2} s^2 \right) ds \dots \dots \dots (3)$$

where ϕ is the angle which the vector from the element of the wave under consideration makes with a direction parallel to OY . Since ϕ is always small, we may write approximately $\cos \phi = 1$ and

$\sin \phi = (\xi - s)/f$. The expressions then become, after some further transformations,

$$H = E_x = C_0 \cos k(at - \rho_0) + S_0 \sin k(at - \rho_0) \dots \dots \dots (4)$$

$$E_y = C_1 \cos k(at - \rho_0) + S_1 \sin k(at - \rho_0) \dots \dots \dots (5)$$

where

$$\left. \begin{aligned} C_0 &= \int_{-\sigma}^{\sigma} \cos (ls - ms^2) ds \\ S_0 &= \int_{-\sigma}^{\sigma} -\sin (ls - ms^2) ds \\ C_1 &= \int_{-\sigma}^{\sigma} \frac{\xi - s}{f} \cos (ls - ms^2) ds \\ S_1 &= \int_{-\sigma}^{\sigma} -\frac{\xi - s}{f} \sin (ls - ms^2) ds \\ l &= k\xi/f \text{ and } m = k\eta/2f^2 \end{aligned} \right\} \dots \dots \dots (6)$$

The direction of the time-mean of the vector [E.H.] will determine the direction of the line of flow of energy at the point. Hence if ψ be the angle which the line of flow makes with the direction OY , we have

$$\tan \psi = \frac{\overline{H.E_y}}{\overline{H.E_x}} = \frac{C_0 C_1 + S_0 S_1}{C_0^2 + S_0^2}$$

This is the differential equation of the lines of flow. We may now put the four integrals C_0 , C_1 , S_0 and S_1 in a form suitable for calculation, expressing them in terms of Fresnel integrals or in terms of Lommel Functions. Since the addition of a constant term in the periodic part does not affect the time-mean value of the expressions for E and H , equations (4) and (5) will continue to hold if

$$\begin{aligned} C_0 &= \int_{-\sigma}^{\sigma} \cos m \left(s - \frac{l}{2m} \right)^2 ds = \int_{-\sigma}^{\sigma} \cos m \left(s - \frac{\xi}{\eta} f \right)^2 ds \\ S_0 &= \int_{-\sigma}^{\sigma} \sin m \left(s - \frac{\xi}{\eta} f \right)^2 ds \end{aligned}$$

$$C_1 = \int_{-\sigma}^{\sigma} \frac{\xi - s}{f} \cos m \left(s - \frac{\xi}{\eta} f \right)^2 ds$$

$$S_1 = \int_{-\sigma}^{\sigma} \frac{\xi - s}{f} \sin m \left(s - \frac{\xi}{\eta} f \right)^2 ds$$

Transforming these further, we have

$$C_0 = \int_{-\left(\sigma + \frac{\xi}{\eta} f\right)}^{-\sigma + \frac{\xi}{\eta} f} \cos mx^2 dx$$

$$= \frac{1}{2} \sqrt{\frac{\pi}{2m}} \left[\int_0^{mf^2 \left(\frac{\sigma}{f} - \frac{\xi}{\eta}\right)^2} J_{-\frac{1}{2}} dy + \int_0^{mf^2 \left(\frac{\sigma}{f} + \frac{\xi}{\eta}\right)^2} J_{-\frac{1}{2}} dy \right] \dots\dots (7)$$

$$S_0 = \frac{1}{2} \sqrt{\frac{\pi}{2m}} \left[\int_0^{mf^2 \left(\frac{\sigma}{f} - \frac{\xi}{\eta}\right)^2} J_{\frac{1}{2}} dy + \int_0^{mf^2 \left(\frac{\sigma}{f} + \frac{\xi}{\eta}\right)^2} J_{\frac{1}{2}} dy \right] \dots\dots (8)$$

And

$$C_1 = \left(\frac{\xi}{f} - \frac{\xi}{\eta} \right) C_0 - \int_{-\sigma}^{\sigma} \frac{1}{f} \left(s - \frac{\xi}{\eta} f \right) \cos m \left(s - \frac{\xi}{\eta} f \right)^2 ds$$

$$= \left(\frac{\xi}{f} - \frac{\xi}{\eta} \right) C_0 - \int_{-f \left(\frac{\sigma}{f} + \frac{\xi}{\eta} \right)}^f \frac{x}{f} \cos mx^2 dx$$

$$= \left(\frac{\xi}{f} - \frac{\xi}{\eta} \right) C_0 + \frac{1}{mf} \sin \frac{k\xi\sigma}{f} \cos \frac{k\eta}{2} \left(\frac{\sigma^2}{f^2} + \frac{\xi^2}{\eta^2} \right) \dots\dots\dots (9)$$

Similarly

$$S_1 = + \left(\frac{\xi}{f} - \frac{\xi}{\eta} \right) S_0 + \frac{1}{mf} \sin \frac{k\xi\sigma}{f} \sin \frac{k\eta}{2} \left(\frac{\sigma^2}{f^2} + \frac{\xi^2}{\eta^2} \right) \dots\dots\dots (10)$$

Hence

$$\tan \psi = \frac{\xi}{f} = \frac{\xi}{\eta} + \frac{2f}{k\eta} \sin \frac{k\xi\sigma}{f} \frac{C_0 \cos \chi + S_0 \sin \chi}{C_0^2 + S_0^2}$$

$$= \frac{\xi}{f} - \frac{\xi}{\eta} + \frac{\sqrt{2}}{\pi} \sqrt{\frac{\lambda}{\eta}} \sin \frac{k\xi\sigma}{f} \cdot \frac{I_1 \cos \chi + I_2 \sin \chi}{I_1^2 + I_2^2} \chi \dots\dots\dots (11)$$

where

$$\chi = \frac{k\eta}{2} \left(\frac{\sigma^2}{f^2} + \frac{\xi^2}{\eta^2} \right)$$

$$I_1 = \int_0^{mf^2 \left(\frac{\sigma}{f} - \frac{\xi}{\eta} \right)^2} J_{-\frac{1}{2}} dy + \int_0^{mf^2 \left(\frac{\sigma}{f} + \frac{\xi}{\eta} \right)^2} J_{-\frac{1}{2}} dy \dots\dots\dots (12)$$

$$I_2 = \int_0^{mf^2 \left(\frac{\sigma}{f} - \frac{\xi}{\eta} \right)^2} J_{\frac{1}{2}} dy + \int_0^{mf^2 \left(\frac{\sigma}{f} + \frac{\xi}{\eta} \right)^2} J_{\frac{1}{2}} dy \dots\dots\dots (13)$$

The values of the integrals $\int J_{-\frac{1}{2}} dy$ and $\int J_{\frac{1}{2}} dy$ have been tabulated by Lommel in his memoir, and using these we can determine the direction of the line of flow at any point. If we integrate

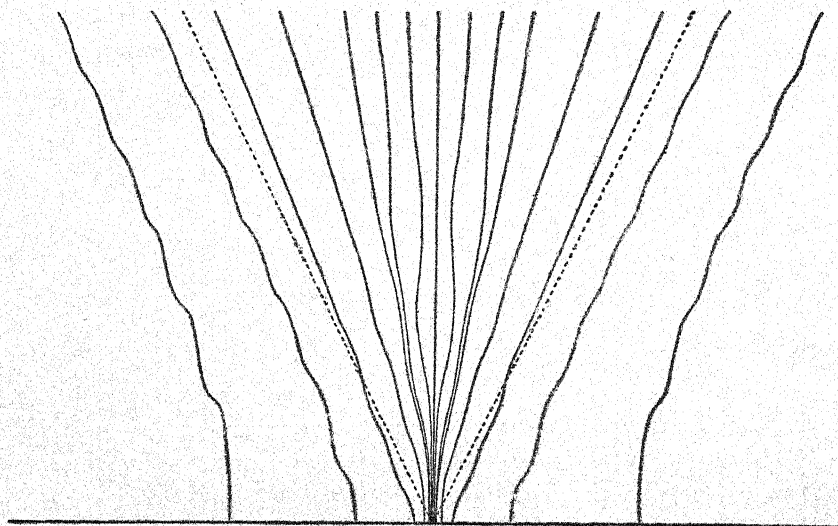


FIG. 2.

the differential equation (11) we can of course get the actual lines of flow in different parts of the field. But it is seen that the equation is not in an easily integrable form, nor can any approxi-

mation be made, in the part of the field which is of most interest, i.e. near the focus, to reduce the equation to an easily integrable form. The lines have therefore to be drawn directly from the differential equation by calculating the direction of flow at successive points and tracing out the lines bit by bit. The curves in the accompanying *plate* have been drawn thus, the direction of a line being determined at successive intervals of $\frac{\pi}{10}$ in the value of $(k\xi\sigma/f)$.

There is also another expression in terms of Lommel Functions which is suitable for numerical calculation at a distance from the focus. Starting from equations (6), we have

$$C_0 = \int_{-\sigma}^{\sigma} \cos(ls - ms^2) ds = 2 \int_0^{\sigma} \cos ls \cos ms^2 ds = Z \text{ (say)} \dots (14)$$

$$S_0 = \int_{-\sigma}^{\sigma} -\sin(ls - ms^2) ds = 2 \int_0^{\sigma} \cos ls \sin ms^2 ds = 2 Z^1 \text{ (say)} \dots (15)$$

$$\begin{aligned} C_1 &= \int_{-\sigma}^{\sigma} \frac{\xi - s}{f} \cos(ls - ms^2) ds \\ &= \frac{2\xi}{f} Z - \frac{2}{f} \int_0^{\sigma} s \sin ls \sin ms^2 ds \\ &= \frac{2}{f} Z \left(\xi - \frac{l}{2m} \right) + \frac{1}{mf} \sin l\sigma \cos m\sigma^2 \dots \dots \dots (16) \end{aligned}$$

Similarly

$$S_1 = \frac{2}{f} Z^1 \left(\xi - \frac{l}{2m} \right) + \frac{1}{mf} \sin l\sigma \sin m\sigma^2 \dots \dots \dots (17)$$

so that

$$\tan \psi = \left(\frac{\xi}{f} - \frac{l}{2mf} \right) + \frac{1}{2mf} \sin l\sigma \frac{Z \cos m\sigma^2 + Z' \sin m\sigma^2}{Z^2 + Z'^2} \dots \dots \dots (18)$$

or since $l = k\xi/f$ and $m = k\eta/2f^2$

$$\tan \psi = \frac{\xi}{f} - \frac{\xi}{\eta} + \frac{f}{k\eta} \sin l\sigma \frac{Z \cos m\sigma^2 + Z' \sin m\sigma^2}{Z^2 + Z'^2} \dots \dots \dots (19)$$

We may now proceed to express Z, Z' in terms of Bessel's functions.

$$\begin{aligned}
 Z &= \int_0^{\sigma} \cos ls \cos ms^2 ds = \int_0^{\sigma} \left(\frac{\pi ls}{2}\right)^{\frac{1}{2}} J_{-\frac{1}{2}}(ls) \cos ms^2 ds \\
 &= \left(\frac{\pi}{4m}\right)^{\frac{1}{2}} \left\{ U_{\frac{1}{2}} \cos m\sigma^2 + U_{3/2} \sin m\sigma^2 \right\} \dots\dots\dots (20)
 \end{aligned}$$

where $U_{\frac{1}{2}}, U_{3/2}$ are Lommel functions defined by

$$U_n = \sum_{s=0}^{\infty} (-1)^s \left(\frac{2m\sigma}{l}\right)^{n+2s} J_{n+2s} \dots\dots\dots (21)$$

Similarly

$$Z' = \left(\frac{\pi}{4m}\right)^{\frac{1}{2}} \left\{ U_{\frac{1}{2}} \sin m\sigma^2 - U_{3/2} \cos m\sigma^2 \right\} \dots\dots\dots (22)$$

Hence

$$\begin{aligned}
 \tan \psi &= \frac{\xi}{f} - \frac{\xi}{\eta} + \frac{f}{k\eta} \sqrt{\frac{4m}{\pi}} \sin l\sigma \frac{U_{\frac{1}{2}}}{U_{\frac{1}{2}}^2 + U_{3/2}^2} \\
 &= \frac{\xi}{f} - \frac{\xi}{\eta} + \frac{\xi}{\eta} \sin x \cdot \frac{P_1 \sin x + Q_1 \cos x}{(P_1 \sin x + Q_1 \cos x)^2 + (P_2 \sin x + Q_2 \cos x)^2}
 \end{aligned}$$

where $x = k\xi\sigma/f$

$$P_1 = 1 - a^2 \left(\frac{3}{x^2} - 1 \right) + a^4 \left(\frac{105}{x^4} - \frac{45}{x^2} + 1 \right) - \dots\dots\dots$$

$$Q_1 = a^2 \cdot \frac{3}{x} - a^4 \left(\frac{105}{x^3} - \frac{10}{x} \right) + \dots\dots\dots$$

$$P_2 = \frac{a}{x} - a^3 \left(\frac{15}{x^3} - \frac{6}{x} \right) + a^5 (\dots) - \dots\dots\dots$$


$$Q_2 = -a + a^3 \left(\frac{15}{x^2} - 1 \right) - a^5 (\dots)$$

$$a \text{ being written for } \left(\frac{\eta}{\xi} \cdot \frac{\sigma}{f} \right).$$

The expressions are obtained by expanding the Lommel functions in a series of Bessel functions and then substituting semi-convergent expansions for the latter (see Gray and Mathews, Bessel Function, p. 42). The expansions are useful as the terms converge rapidly in regions well outside the geometric cone, a being then small.

Some principal features of the lines of flow as drawn from the above calculations may now be noted. The first thing that strikes us is the remarkable concentration of the lines when approaching the focus. Most of the lines of flow originating

within the geometrical cone of light (represented by the dotted lines in the Plate) from the aperture crinkle down to the central maximum in the focal plane, a fact which is but to be expected in view of the large concentration of light that is known both experimentally and from the elementary theory of diffraction to occur at the focus. Secondly, the lines of flow which reach down to a point at a distance from the focus originate near the boundary of the aperture, and this perhaps explains the luminosity of the boundary of the aperture when viewed from behind a small obstacle covering the focus. The lines of flow are practically straight lines radiating to the focus, except in the close neighbourhood of the focus itself where the crinkling is much more pronounced. The "visibility" of the fringes observed in any plane depends on the amplitude of the crinkles, and hence increases as the plane of observation is moved down towards the focal plane and the Fresnel class is gradually transformed to the Fraunhofer class of diffraction phenomena.



IV. On some Indian Stringed Instruments.

By C. V. Raman, M.A., D.Sc. (Hon.) Palit Professor of
Physics in the Calcutta University.

(Plate I.)

CONTENTS.

- I.—Introduction.
- II.—The Form of the Bridge in the “Tanpura” and the “Vina.”
- III.—The Failure of the Young-Helmholtz Law.
- IV.—Outline of Mechanical Theory.
- V.—Summary.

I. INTRODUCTION.

A fascinating field for research offers itself in the scientific study of the numerous kinds of musical instruments to be found in India. Some of these instruments of indigenous origin are of undoubted antiquity and disclose a remarkable appreciation of acoustic principles. An investigation of their special features in comparison with those of instruments of other countries may be expected to yield results of great interest. An instance of the fruitfulness of the line of work here suggested is to be found in the present author's research on the Indian Musical Drums, which have been found to embody in a practical form the solution of the problem of loading a circular drumhead in such a manner as to make it give a harmonic succession of overtones in the same way as a stringed instrument.¹ In the present paper it is proposed to offer a preliminary note on the results of the author's study of some Indian stringed instruments.

¹ See ‘Nature’ Feb. 8, 1920. A fuller account of the work on these musical drums is shortly to be published as a Bulletin of this Association.

2. THE FORM OF THE BRIDGE IN THE "TANPURA" AND THE "VINA."

The "Tanpura" and the "Vina" are two of the most highly valued indigenous stringed instruments intended to be excited by plucking. Plate I, Fig. 1 illustrates the form of the "Tanpura." This instrument has no frets and is intended merely to be used as a drone in accompaniment with vocal or other music. It has four metal "strings" which are stretched over a large resonant body and can be accurately tuned up to the right pitch by a simple device for continuous adjustment of tension. The remarkable feature of the "Tanpura" to which I wish to draw attention is the special form of bridge fixed to the resonant body over which the strings pass. The strings do not come clear off the edge of a sharp bridge as in European stringed instruments, but pass over a curved wooden surface fixed to the body which forms the bridge. The exact length of the string which actually touches the upper



FIG. 3.—The Bridge of the Tanpura.

surface of the bridge is adjusted by slipping in a woollen or silken thread of suitable thickness between each string and the bridge below it and adjusting its position by trial. Generally the thread is moved forwards or backwards to such a position that the metal "string" just grazes the surface of the bridge. The description will be clearer on a reference to Fig. 3 above where the bridge and the string passing over it are indicated diagrammatically.

The "Vina" on the other hand is a fretted instrument intended for use in playing melodies (Fig. 2 in Plate I). The form of the bridge adopted in it differs from that of the "Tanpura" in two respects. The upper curved surface of the bridge in the "Vina" is of metal, and the special mode of adjustment of contact by means of a thread used in the "Tanpura" is dispensed with, and the string merely comes off the curved upper surface of the bridge at a tangent, as indicated diagrammatically in Fig. 4.

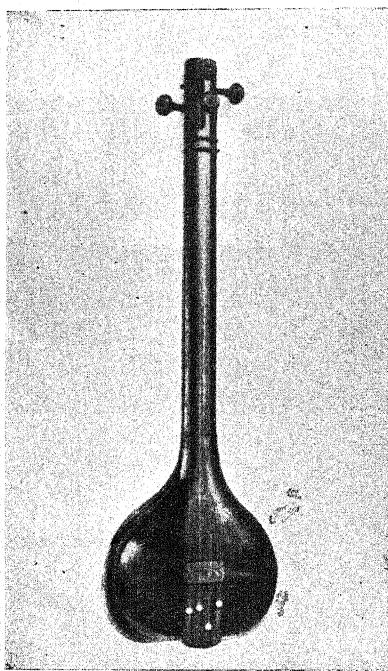


Fig. 1 Tanpura

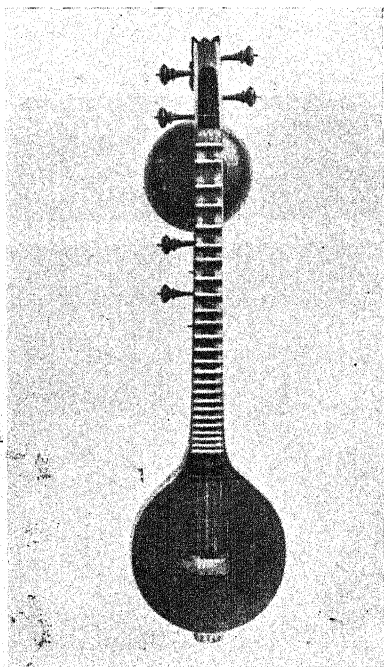
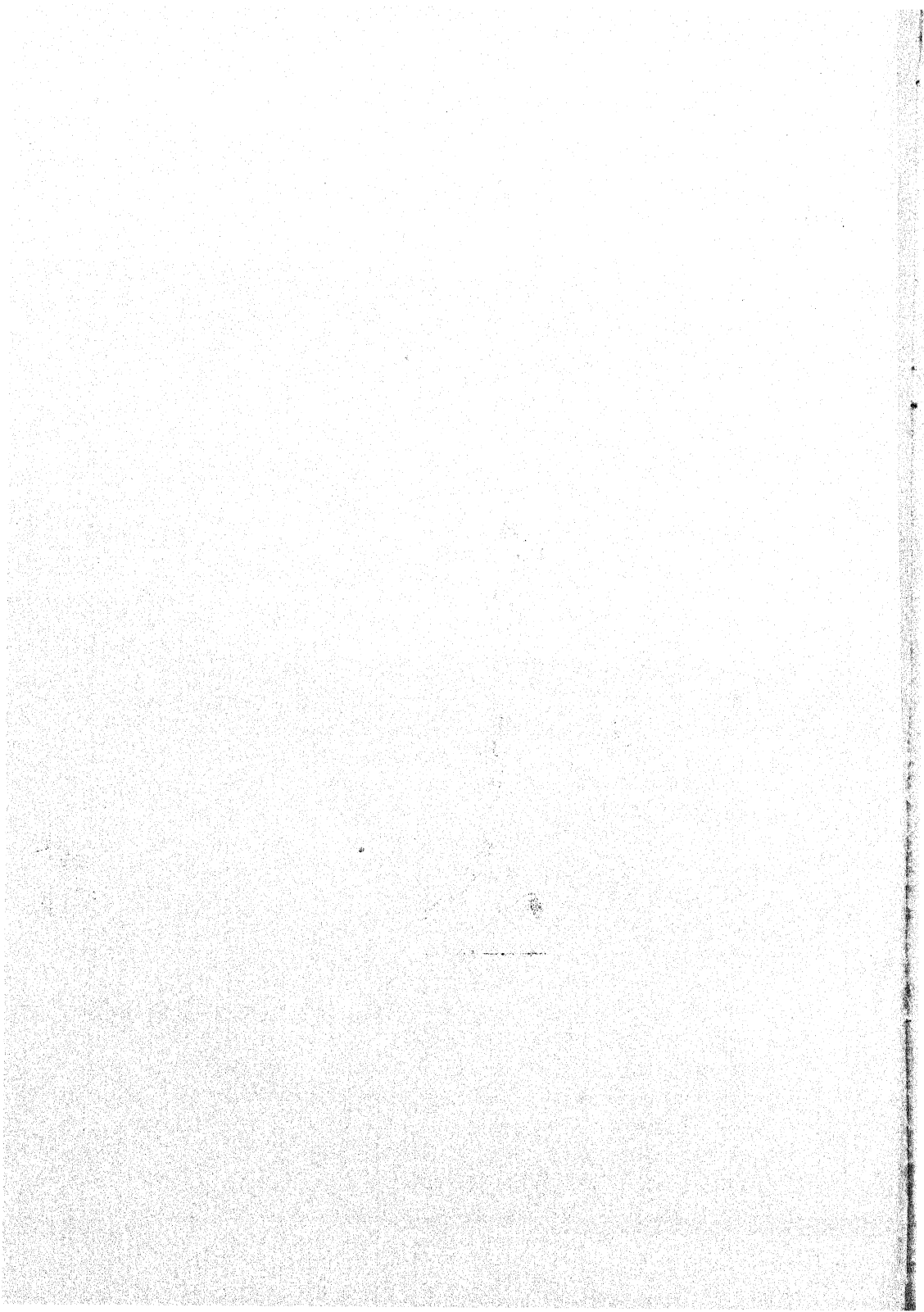


Fig. 2 Vina



(No attempt is made in this figure to indicate the exact form of the lower part of the bridge.)

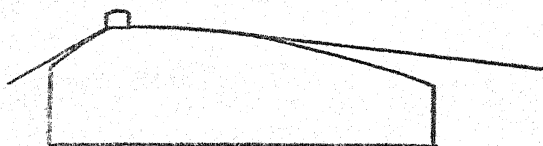


FIG. 4.--The Bridge of the Vina.

The bridge of the "Vina" is also much higher above the body of the instrument than in the "Tanpura." Even when the strings are pressed down on the frets when the instrument is being played, the curvature of the upper surface of the bridge ensures the string always leaving the bridge at a tangent to it as shown.

3. THE FAILURE OF THE YOUNG-HELMHOLTZ LAW.

The special form of bridge illustrated above has a very remarkable influence on the tone-quality. This can be most readily demonstrated in the "Tanpura." When the adjustment of contact of string and bridge is made carefully by trial, the instrument is highly sonorous, giving a tone of fine musical quality. If on the other hand the grazing contact of string and bridge is rendered inoperative (as for instance by inserting a small piece of metal between the string and the surface of the bridge)" the tone becomes dull and insipid. A similar remark applies also to the case of the "Vina," though the difference is less striking in the latter case.

In attempting to find an explanation for the difference in tone-quality produced by the special form of bridge, the author made a surprising observation, namely, that in the tone of the "Tanpura" or the "Vina," overtones may be heard powerfully which according to known acoustical principles should have been entirely absent. According to the law enunciated by Young and Helmholtz, if the string is plucked at a point of aliquot division, the harmonics having a node at the point of excitation should be entirely absent. This law may be readily verified on an ordinary sonometer with the usual form of bridge. For this purpose, the position of the node should first be found exactly by trial by putting the finger in contact with the string and plucking else-

where so as to elicit the overtones desired. Having found the position of the node, the string should be plucked exactly at that point and then again touched with the finger *at the same point*. On an ordinary sonometer, this results in the sound being immediately quenched inasmuch as the finger damps out all the partials except those having a node at the point touched, and the latter are not excited in the first instance in accordance with the Young-Helmholtz law. On trying the same experiment with the "Vina" or the "Tanpura", it will be found that the overtone having a node at the plucked point sings out powerfully. In fact the position of the plucked point hardly appears to make a difference in regard to the intensity of the overtones in the "Tanpura." This remarkable result is not due to any indefiniteness in the position of the node point, as the latter is found to be quite well defined as is shown by the fact that in order to demonstrate the effect successfully, the string must be plucked and then touched exactly at that right point, otherwise the sound is quenched. We are thus forced to the conclusion that the effect of the special form of bridge is completely to set aside the validity of the Young-Helmholtz law and actually to manufacture a powerful sequence of overtones including those which ought not to have been elicited according to that law.

4. OUTLINE OF A MECHANICAL THEORY.

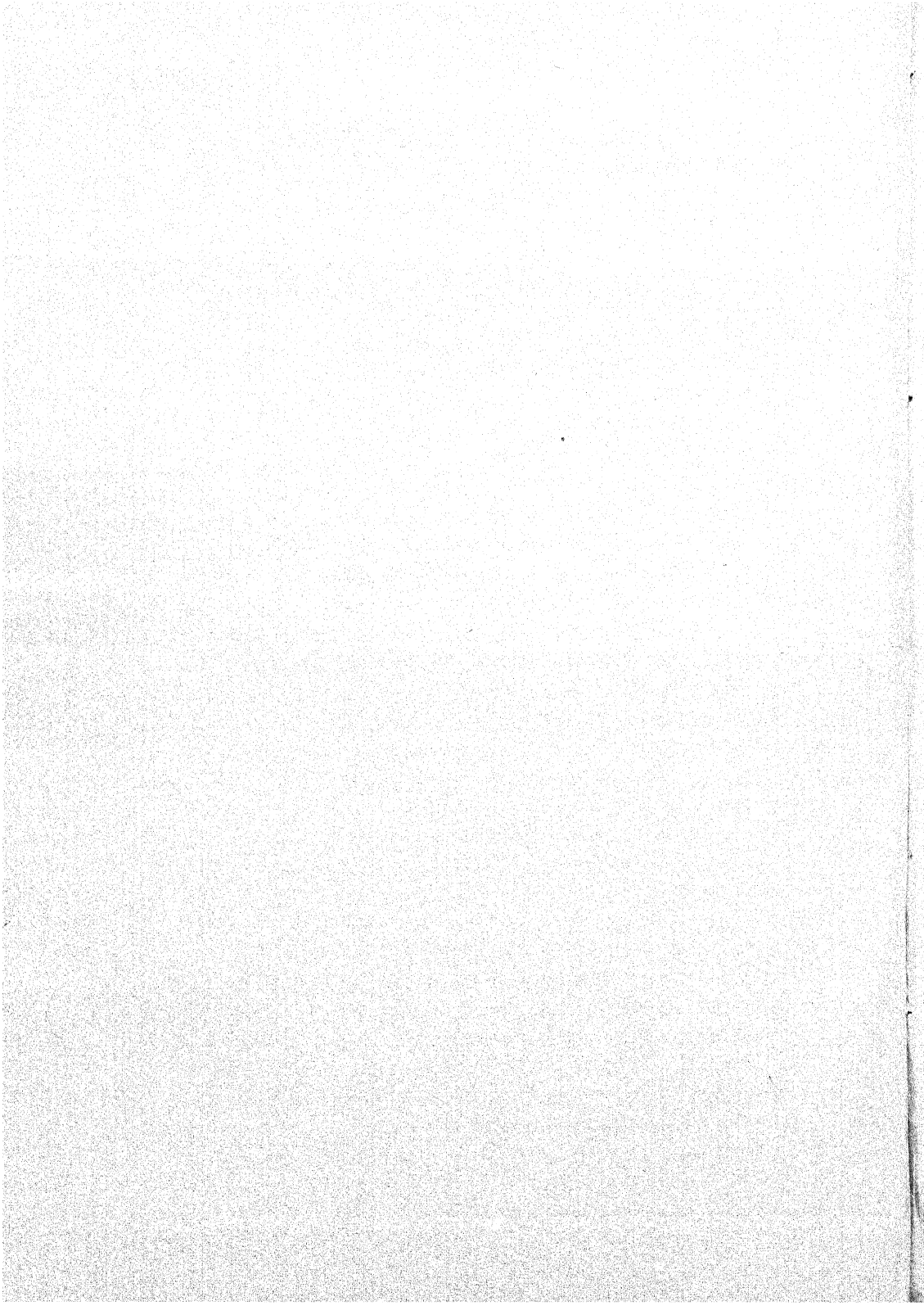
Some photographs of the vibration-curves of a "Tanpura" string were made at the suggestion of the author by Mr. Ahmed Shah Bukhari at the Government College, Lahore, last November. They showed that in consequence of the grazing contact at the bridge, the vibration of the string decreased in amplitude and altered its form at a much more rapid rate than when the grazing contact was rendered ineffective. A more complete investigation is obviously desirable. From first principles, however, it is obvious that in the 'Tanpura' the forces exerted by the vibrating string on the bridge must be very different from what they would be for a bridge of ordinary form. It seems probable that by far the greater portion of the communication of energy to the bridge occurs at or near the point of grazing contact. The forces exerted by the string on the bridge near this point are probably in the nature of impulses occurring once in each vibration of the string.

This would explain the powerful retinue of overtones including even those absent initially in the vibration of the string. At a slightly later stage, the reaction of the bridge on the string would result in a modification of the vibration form of the latter and bring into existence partials absent initially in it. There would in fact be a continual transformation of the energy of vibration of the fundamental vibration into the overtones.

The foregoing explanation of the character of the tones of the "Tanpura" would not be fully applicable to the "Vina" as the forces exerted by the string on the bridge in this case would not be purely of an impulsive character. There is however a certain portion of the bridge over which the string comes into intermittent contact during the vibration, and it seems very probable that the theory for this case is intermediate in character between that for the 'Tanpura' and those for stringed instruments with bridges of the ordinary type. Further experimental work is needed in support of this view.

5. SUMMARY.

The present paper deals with the remarkable acoustic properties of the "Tanpura" and the "Vina" which are two of the most highly reputed among Indian stringed instruments. The form of the bridge used in these instruments is quite different from that usually found in European stringed instruments. In the 'Tanpura' the string passes over the wooden upper surface of the bridge which is curved to shape, and by insertion of a thread of wool or silk, a finely adjustable grazing contact of string and bridge is secured. In the 'Vina' the upper surface of the bridge is of curved metal and the string leaves it at a tangent. The tones of these instruments show a remarkable, powerful series of overtones which gives them a bright and pleasing quality. Experiment with these instruments shows that the validity of the Young-Helmholtz law according to which partials having a node at the plucked point should not be excited is completely set aside. A possible mechanical explanation of this result is suggested.



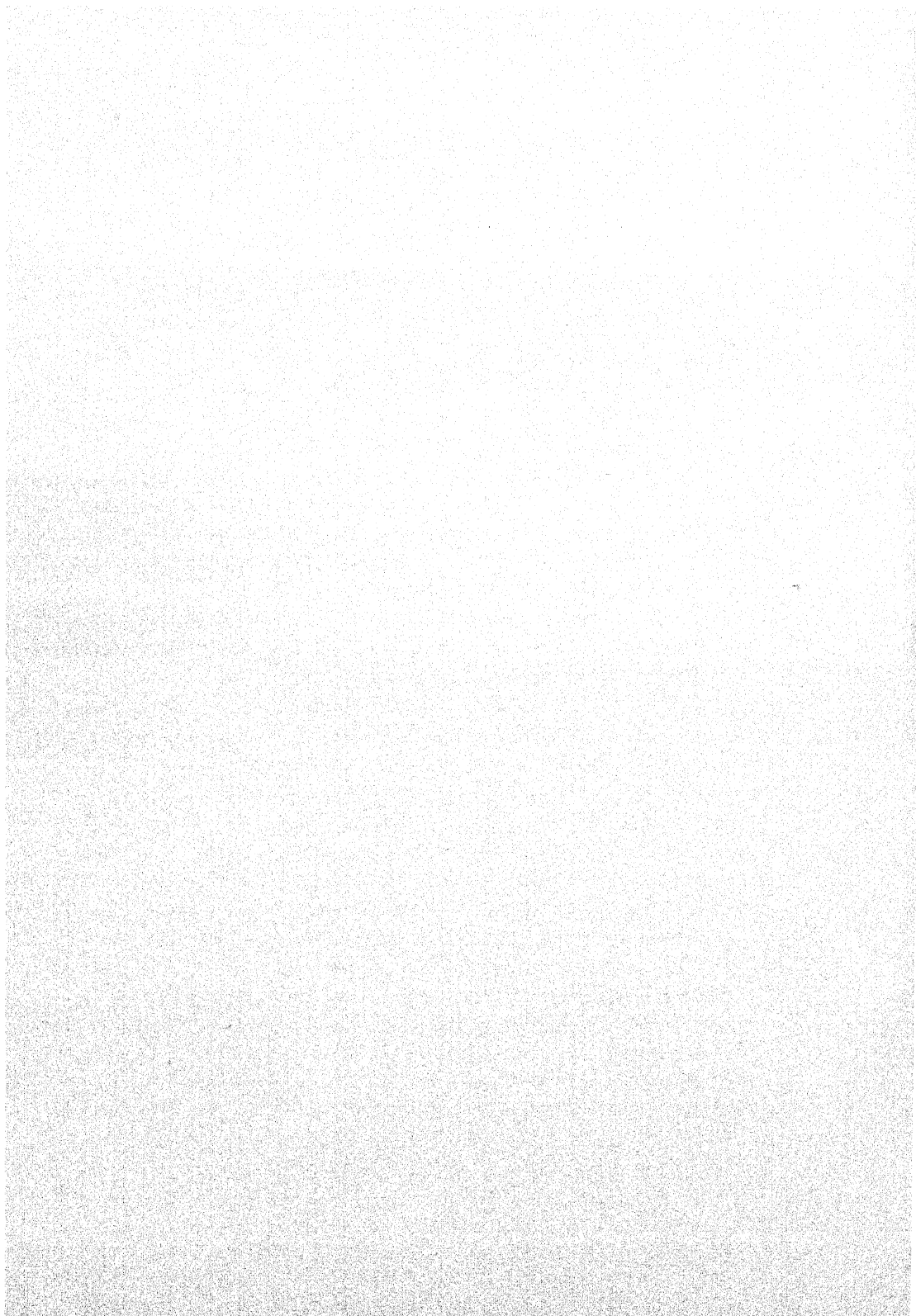
V. Note on the Omori-Ewing Seismograph.

By Dr. Gilbert T. Walker, C.S., Sc.D., F.R.S.I.

[Extract from a letter to Prof. C. V. Raman with reference to the preceding paper.]

In connection with your very interesting stringed instrument it may be convenient to have a more definite statement regarding the apparently somewhat similar effect observable with seismographs.

As you probably know in the Omori-Ewing pattern the heavy weight, which forms part of a horizontal pendulum with a period of say 40 seconds, is prevented from moving too far and perhaps breaking the instrument during a violent earthquake by stops: and it has been noticed that when owing to faulty adjustment the heavy weight is in contact with one of these the amplitude of the resulting record is greatly magnified. I think the explanation is fairly simple. If the ground has a harmonic motion of total amplitude $2a$ and frequency m , the period being of the order of two seconds, then the maximum velocity will be am . Let us suppose that this motion occurs only for a time π/m which will be comparable with one second. We suppose as the simplest possible case that the stop and the weight, being initially in equilibrium, are then in contact, and that the stop, owing to the motion of the ground, suddenly starts away from the weight with velocity am : on its return to the weight it will suddenly impart to it this velocity, and the ground then is supposed to come to rest. The total amplitude $2b$ of the swing of the weight will be connected with its natural frequency n by the equation $bn=am$. Thus the amplitude of the swing of the weight after being struck by the stop bears to what it would have been if the stop had not been there the ratio b/a or m/n , which is comparable with 20. Of course in practice the motion is complicated by the motion of the column from which the instrument is suspended and by subsequent motion of the ground; but as a first approximation I regard the result as correct. The motion of the weight is magnified on the trace but the ratio of the amplitudes on the trace corresponding to contact with the stop and the normal adjustment is of course still b/a .



VI. Some Observations on Interference Phenomena in Non-Homogeneous Light.

By Nihal Karan Sethi, D.Sc., Assistant Professor of Physics
in the Benares Hindu University.

CONTENTS.

SECTION I.—Dispersion of White-Light Fringes by a Retarding Plate.

SECTION II.—Dispersion of White-Light Fringes in a Wedge-shaped Plate
by observation through a Prism.

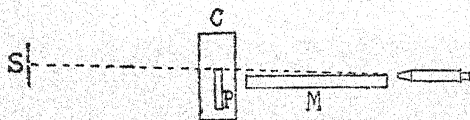
SECTION III.—Summary.

SECTION I.—DISPERSION OF WHITE-LIGHT FRINGES BY A RETARDING PLATE.

It is well known that when a thin plate of some transparent substance is introduced in the path of one of two interfering beams, e.g. in Lloyd's single mirror experiment, the effect is to shift the interference fringes by n fringe-widths towards the path of the beam which is retarded, where $n = (\mu - 1) \frac{D}{\lambda}$ and D is the thickness of the plate. And in the case of illumination with white light also, in spite of the dispersion of the plate, the system of fringes appears to shift as a whole without any change in their number or appearance except that as found by Airy and Stokes, the central or achromatic band shifts to a slightly greater extent than might be expected from the foregoing expression. Its achromatism also does not remain as perfect as before the introduction of the plate. The present writer has, however, found a very simple but interesting case in which the introduction of a retarding plate completely abolishes the achromatic band and what is more increases the number of fringes observable with white light to a very considerable extent. And still more surprising is the fact that any attempt at partially monochromatising

the light by interposing coloured screens in its path actually diminishes instead of increasing the number of visible fringes.

To perform the experiment, Lloyd's fringes are first obtained in the usual manner and a microscope carefully focussed on them. In front of the mirror *M* (Fig. 1) which forms these fringes is placed a cell *c* containing a mixture of carbon bisulphide and benzene, the composition of which has been adjusted till its refractive index is equal to that of the glass plate for the blue end of the spectrum. The retarding plate *P* which is a plate of glass about a millimeter or two or even 5 mm. in thickness is immersed in this cell in such a manner that one of the two interfering beams passes through it while the other passes through the liquid mixture only. Further, the glass plate should be on the same side as the mirror producing the fringes. When the position of this plate has been carefully adjusted by shifting the cell, a surprisingly large number of fringes will be observed in the field of view, and they will all appear to be coloured. Start-



ing from near the edge of the plate (as seen in the microscope) the brighter fringes will be found to gradually change their colour from red to orange, yellowish white, green and then blue. The darker fringes which are only slightly weaker in intensity, will be found to have a colour more or less complementary to the neighbouring brighter fringes, except in the region in the middle of the field where the colours are hardly noticeable. Seen through a red glass, the fringes towards the edge of the plate appear to gain in contrast but the rest of the fringes vanish altogether. A yellow or green glass shifts the region of maximum visibility to about the middle of the field and a blue-violet glass makes the fringes at the other end distinctly better but completely destroys those towards the edge of the plate. The width of these fringes is also not constant but decreases gradually from the red side of the field towards the blue.

The effect observed here is no doubt of the same nature as the shift of the achromatic band by a retarding plate discussed by

Airy and Stokes, differing from it in that the magnitude of the shift in the present instance differs very largely for different groups in the spectrum. It has been shown by the late Lord Rayleigh¹ that the shifted position of this band is given by the relation

$$R + \left(\mu - 1 - \lambda \frac{d\mu}{d\lambda} \right) D = 0 \dots \dots (A)$$

where D and μ are respectively the thickness and the refractive index of the plate and R is the relative retardation due to other causes. In the usual case, the introduction of the dispersion term $\lambda \frac{d\mu}{d\lambda}$ involves only a small correction; for $\mu - 1$ is relatively much larger and D cannot be increased to any great extent. In the present case, however, this relation becomes

$$R + \left\{ (\mu - \mu') - \lambda \left(\frac{d\mu}{d\lambda} - \frac{d\mu'}{d\lambda} \right) \right\} D = 0 \dots \dots (B)$$

so that the effective value of the first term within the curled brackets has been greatly reduced throughout the spectrum and would actually vanish for the wave-length for which the refractive indices of the plate and liquid are equal, while the second term depending on the difference of the dispersive powers remains finite throughout. At the same time in the present case, the value of D has been considerably increased. The result is that, while usually the regions of maximum visibility in the field for different groups in the spectrum are more or less coincident, they will not be so with the present arrangement, for equation (B) is satisfied for different wave-lengths at points in the field well separated from each other. We thus have what might be called the dispersion of the achromatic band, the groups near the red end of the spectrum showing the fringes best near the edge of the plate and the groups near the blue end relatively far out in the field. The observable fringes are thus spread out over a large extent and we have their number greatly increased.

It is evident that the visibility of the fringes measured in terms of the total intensity of illumination in the bright and dark portions must suffer very considerably in this process, for in that part of the field where, for example, the fringes are best seen

¹ Phil. Mag. Sept. 1904 and Scientific Papers, Vol. V.

in red light, the rest of the spectrum must be producing a fairly strong and more or less uniform illumination. The colour of this uniform illumination will, however, be bluish and the visibility resulting from the contrast of colour will be quite good for we shall have in juxtaposition two such widely different colours as red and bluish green.

Attention may also be drawn to the fact that the experiment described above may be very conveniently used to illustrate the fact that the shift of the region of best visibility is determined by the difference of group-velocity of light in the plate and the liquid and not by the difference of wave-velocity; for if the refractive index of the liquid mixture is adjusted to be equal to that of the glass plate for any specified part of the spectrum, say the green, we may have zero retardation for light of this colour, but the fringes are visible far out in the field and not near the edge of the plate. If, on the other hand, the refractive index of the liquid mixture is so adjusted that the group-velocity for red light is identical in the glass plate and the liquid,¹ the red fringes remain at the edge of the plate and do not shift out, although the glass has now a decidedly greater refractive index than the liquid and hence retards the waves. As the refractive index of the liquid is gradually altered, the side on which the plate should be introduced in order to enable the shifted fringes to be observed with Lloyd's mirror, does not require to be changed at the stage at which the wave-velocities in the two media are equal but as in the case of Powell's bands, at the stage when the group-velocities are identical.

It seems probable that the large increase in the number of white-light fringes in an interferometer and their breaking up into groups observed by R. W. Wood² on introducing sodium vapour into the path of one of the beams is essentially of the same nature as the phenomenon described above. In Wood's experiment, as in the present case, the term $\mu - 1$ is small and

$\lambda \frac{d\mu}{d\lambda}$ becomes relatively much larger specially in the neighbourhood of the sodium absorption band. The achromatic fringe is,

¹ See a paper by the present author on "Powell's bands and the Group-velocity of light in dispersive media," *Phy. Rev.* Dec. 1920, p. 519.

² *Phil. Mag.* Sept. 1904, and Wood's *Physical Optics*, second edition, p. 141.

therefore, dispersed over a considerable part of the field and in opposite directions on the two sides of the sodium line resulting in the visible fringes breaking up into groups and in an increase of their number. The latter effect is not fully explained in Wood's paper and in the appendix to it by the late Lord Rayleigh.

A similar increase in the number of fringes should also be observed on introducing a moderately thick plane-parallel glass plate in the path of one of the interfering beams in a Michelson Interferometer, and adjusting the moveable mirror to bring back the white-light fringes. In this case much greater thickness of the glass plate can be used than in the experiment with Lloyd's fringes, and although $\lambda \frac{d\mu}{d\lambda}$ is small in comparison with $\mu - 1$ of equation (A) above, the increased thickness of the glass plate causes the separation of the different groups of waves. If t is the distance through which the moveable mirror is shifted, it is evident that the maximum visibility for a group of waves with wave-length λ will be obtained when

$$t - \left\{ (\mu - 1) - \lambda \frac{d\mu}{d\lambda} \right\} D = 0$$

If t_1 and t_2 are the values of t for red and violet groups respectively,

$$t_2 - t_1 = \left\{ (\mu_2 - \mu_1) - \left(\lambda_2 \frac{d\mu}{d\lambda_2} - \lambda_1 \frac{d\mu}{d\lambda_1} \right) \right\} D.$$

And if $\mu = A + \frac{B}{\lambda^2}$,

$$t_2 - t_1 = 3 B D \cdot \left\{ \frac{1}{\lambda_2^2} - \frac{1}{\lambda_1^2} \right\}.$$

With $D = .1$ c.m. and $B = .5 \lambda 10^{-10}$, this will give $t_2 - t_1 = .003$ cm. approximately. This is equivalent to

$$\frac{2 \times .003}{6 \times 10^{-5}} = 100 \text{ wave-lengths.}$$

So that in this case about a hundred fringes should be visible. It is also evident that they will exhibit the same colour-sequence and the same variation in width as in the case of Lloyd's fringes dealt with above. Unfortunately, however, for want of a plane-parallel glass plate of suitable thickness, the writer has not yet been in a position to verify this result.

SECTION II.—DISPERSION OF WHITE-LIGHT FRINGES IN
A WEDGE-SHAPED PLATE BY OBSERVATION
THROUGH A PRISM.

The peculiar dispersion of the achromatic band described in the previous section led the writer to carefully examine whether the same phenomenon is not taking place in the case when the white-light interference fringes formed by reflection from a 'thin plate' are viewed through a prism. Newton had found that in this case, not only do the fringes appear in that part of the field where without the prism mutual overlapping of the different colours had produced uniform illumination but that their number is greatly enhanced and he sometimes estimated this to be more than a hundred.¹ The formation of these visible bands at unusually large thicknesses of the plate is easily understood, for it depends indeed upon precisely the same principles as the shift of the achromatic band produced by a retarding plate. But the explanation of the increased number is not so obvious, more specially in the case of a thin plate bounded by plane surfaces inclined to one another at a small angle. Referring to this wedge-shaped plate, the late Lord Rayleigh remarks² that even when viewed through a prism, the succession of colours in white light and the number of perceptible bands should remain much as usual, because unless the fringe-width changes from place to place, no relative shifts of the various colours can anywhere make the widths of the bands the same for all parts of the spectrum—a condition absolutely necessary for the formation of a *truly achromatic system* of fringes. He, therefore attributes this phenomenon observed by Newton to the curvature of the surfaces of the plate and the consequent non-uniformity of the angle between them.

But it would appear from the experiments described below that this view of the case is hardly satisfactory, for even in the case of a wedge bounded by the plane surfaces of two interferometer plates, we do actually observe a considerable increase in the number of fringes. Although there is no particular difficulty in seeing these fringes in the usual manner, it is much more convenient to mount the wedge at the focus of a collimating

¹ See quotation on page 311, Rayleigh's Sc. Papers, Vol. III.

² Scientific Papers, Vol. III, p. 313.

lens and to view the fringes by means of a telescope. The prism is placed in front of the telescope and by the suitably adjusting the inclination of the prism and the focus of the telescope, a large number of fringes are brought into the field of view. The refracting edge of the prism should be towards the thicker side of the plate.

It must be admitted that these fringes do not by any means constitute a completely achromatic system of bands and perhaps Lord Rayleigh may be quite right in denying the existence of such a system in the case of a true wedge.¹ But all the same, there is in the present case a real increase in the number of perceptible fringes. The colour distribution was found to be exactly similar to what has been described in the case considered in the previous section and it was evident that here also there must be the same dispersion of the achromatic band. The width of the bands too, was not uniform and as was expected it decreased slowly from the red to the blue side.

To test the matter further, the slit of a direct vision spectroscopic was placed at the focus of the observing telescope and the spectrum of the successive bands was examined. It was found to be crossed by numerous dark bands except for a narrow region free from bands on either side of which the dark bands gradually crowded in on each other. This region in the spectrum accordingly represents a group for which the visibility of the fringes is best in the part of the field under observation. As the spectroscopic was moved along the bands, it was found that the position of this group slowly shifted from one end of the spectrum to the other. But probably a better idea of this is obtained when the spectroscopic is turned round and its slit placed across the bands. We have in this case the various spectra of the successive bands arranged side by side and a series of beautiful curved bands appear in the spectrum, by observation of which it becomes evident at once that the region in the field at which the fringes are stationary and the visibility is accordingly best, is different for different groups in the spectrum.

That the region of best visibility of the fringes is dispersed to

¹. Reference may be made to the investigation of Mascart (*Traité de Optique*, Tome I, p. 432). The change in the apparent width of the fringes seen through the prism is a factor to be also taken into consideration.

different parts of the field can be easily seen on investigating the effect of the prism by exactly the same method as used by Rayleigh¹ in the case of a plate with cylindrical surfaces. If we choose the point of contact as the origin of x , the thickness of the wedge may be taken to be

$$t = bx$$

where b depends on the inclination of the surfaces. The black of the n^{th} order for wave-length λ occurs when

$$\frac{1}{2}n\lambda = bx$$

$$\text{or } x = \frac{n}{2b}\lambda$$

$$\text{So that } \frac{dx}{d\lambda} = \frac{n}{2b}$$

The n^{th} band formed actually at x is seen displaced under the action of the prism. The amount of the linear displacement z is proportional to the distance D at which the prism is held, so that we may take approximately

$$\frac{dz}{d\lambda} = -\beta D,$$

β representing the dispersive power of the prism. The condition that the n^{th} band is stationary for small variations of λ , is accordingly

$$\frac{d(x+z)}{d\lambda} = 0$$

$$\text{i.e. } \frac{n}{2b} = \beta D$$

$$\text{or } x = 2b\beta D.$$

Now, in the case of a prism taking Cauchy's law of dispersion, β is proportional to λ^{-3} , so that we can put $\beta = \frac{2M}{\lambda^3}$.

$$\therefore n = \frac{4bMD}{\lambda^3}$$

and consequently $x = \frac{2MD}{\lambda^2}$.

It is thus seen that the region of best visibility of the fringes is different for different portions of the spectrum; for example, if

¹ Scientific Papers, Vol. V, p. 427.

for red light in any case $n=20$, the value of n would be 160 for the violet and we should be able to see about a hundred fringes.

An objection might be raised that all the fringes obtained in the above manner are coloured whereas according to Newton, where they seemed most distinct, the bands were only black and white successively without any other colours intermixed, though, in other places, they were undoubtedly coloured. But there is no real difficulty in explaining this apparent contradiction, if we remember that where they seem most distinct, the achromatisation has taken place for the brightest central portion (greenish yellow) of the spectrum. The colour of the rest of spectrum which forms a back-ground of uniform illumination for these fringes is not different from the colour of the maxima and what the eye perceives in this part of the field is merely the fluctuation of the intensity. In this connection it may be useful to recall the observation of Lord Rayleigh with reference to the achromatisation of Lloyd's fringes by a prism for the yellow-green part of the spectrum, where he says, "It is remarkable that so little colour should be apparent, on direct inspection of the bands." "It would seem," he adds, "that the eye is but little sensitive to colours thus presented, perhaps on account of its own want of achromatism."¹

III.—SUMMARY.

1. When a retarding plate is introduced in the path of one of two interfering beams, e.g. in Lloyd's experiment, the interference fringes formed in white light usually appear to shift as a whole without any change in their number or appearance, except that for the central or achromatic band the shift is slightly greater. If, however, the retarding plate is one of glass immersed in a liquid of nearly equal refractive index, these fringes are found to increase considerably in number and any attempt at partially monochromatising the light by interposing coloured screens actually *reduces* instead of increasing this number. There is left no band which may even approximately be described as central and the width of the fringes too is not constant but varies continuously from one end of the field to the other. This is shown to be due to what may be called the dispersion of the achromatic band which causes different

¹ Rayleigh, Sci. Papers, Vol. III, p. 300.

groups of waves in the spectrum to produce the maximum visibility of the fringes in different parts of the field.

It is also probable that the increase in the number of white light fringes observed by Wood on introduction of sodium vapour into the path of one of the beams and which does not appear to have been explained, is also caused by a similar dispersion.

2 Newton had observed that the number of white-light fringes seen in a thin plate is considerably increased by viewing them through a prism. This phenomenon was explained by the late Lord Rayleigh as being due to the formation of an achromatic system of fringes, the necessary condition for which being that the thin plate should be bounded by curved surfaces. And according to him, this phenomenon could not be expected in a truly wedge-shaped plate. The present writer has, however, actually observed it even in the case of a plate of air formed between two interferometer plates, and has shown that an achromatic system of fringes is not needed to explain Newton's observations. The dispersion of white-light fringes referred to above is sufficient in this case as well, not only to explain the enhanced number of fringes but also to give a colour distribution which appears to correspond almost exactly with Newton's observation.

3. It is to be noted that both the cases dealt with in this paper serve to emphasise and illustrate the point of view from which interference phenomena in non-homogeneous light ought to be regarded and on which Schuster rightly laid so much stress. That considerable simplicity is gained by thinking in terms of the *groups* of waves and of the *group-velocity* is perhaps best illustrated by the fact that in the case of the retarding plate discussed, the enormous shift of the fringes at the stage of equality of refractive indices of the plate and of the liquid, and the zero shift at a different stage do not require elaborate explanations and it follows as a matter of course that the shift ought to be zero at the stage of equality of group-velocities.

The writer has much pleasure in expressing his best thanks to Prof. C. V. Raman for his valuable suggestions in the course of the work.

CALCUTTA,
The 21st March, 1921.

VII. On the Beating Tones of Singing Flames.

By Durgadas Banerji, M.Sc., Demonstrator in Physics.
University College of Science, Calcutta.

(Plates II and III.)

I.—INTRODUCTION.

The maintenance of steady vibrations of the air-column in a cylindrical tube by the intermittent supply of heat in the well-known experiments with 'Singing flames' has been the subject of several enquiries from the earliest times by Chladni, Faraday, Tyndall and Sondhauss. Lord Rayleigh¹ has given a theory which is sufficient in a general way to explain the experimental facts formulated by Sondhauss.² Coming to the more recent work on the subject, it appears that the phenomena of singing flames are not so simple as might be supposed from the works of the early investigators. Würschmidt³ has studied both theoretically and experimentally the notes produced in the singing tube. The main conclusions arrived at in his paper are that the notes emitted are not those which might have been produced by the surrounding tube treated as a pipe open at both ends, and that the frequency of the various notes depends upon the length of the flame and its position in the tube, conditioned by the unequal distribution of temperature in different parts of the tube. G. Athenesiadis⁴ has described a singing flame which gives out several sounds simultaneously and specially a flame which responds to two tones, one proper to that of the jet and the other depending on the tube used to enclose it. At this stage of double-sound he appears to have

¹ Scientific Papers, Vol. I, p. 350. Vide also Barton's Sound, 1914, p. 351.

² Pogg. Ann. 109, p. 426. ³ Verh. Deutsch. Physik. Gesellschaft, 1916, p. 444.

⁴ Comptes Rendus 146, 1908, pp. 533-534.

observed a decrease in the flame as the result of reinforcement of sound proper to the tube.

While working on this line, the vibrations of the singing flame have been observed to present a novel appearance when the flame length is so adjusted as to be intermediate between the stages when the fundamental of the tube on one side and the first overtone on the other are most prominently brought out. This and the effect of different lengths of the flame on the nature of the maintained motion have been overlooked by Athenesiadis although the conditions of his experiment as regards the orifice and the tube seem to be indentical with that used by the author. The most striking effect is the peculiar throbbing appearance of the flame and the rythmic variation in the intensity of the note emitted at this stage. The subject of the present paper is a study of some of the peculiarities of this form of vibration of the singing flame which gives rise to beating tones.

Recently I have secured an extensive series of photographs of the flame, so as to bring out the rythmic variations in the form of its vibration as well as the different phases in the motion of the components of the flame, under varying conditions of pressure and flame length, some of which are reproduced in Plates II and III. The coal-gas flame which shows the features more beautifully, being not sufficiently actinic, acetylene gas was used instead and gave satisfactory results but for the accumulation of carbon on the nozzle liberated by the burning of the gas which altered to some extent the character of the vibration when the accumulation was too great. This was avoided by occasional breaking off a portion of the tip of the nozzle tube so as to leave a clean bore every time. The photographs reproduced in the paper were taken with a Zeiss lens of focal length 5 cm. on moving photographic plates.

II.—EXPERIMENTAL METHOD AND RESULTS.

A glass tube about a metre long and 2 cms. in internal diameter is fastened vertically on a stand so that its lower end is at a sufficient height above the table. One end of a glass tube $\frac{1}{2}$ cm. in internal diameter and 20 cms. long is drawn out to a fine jet of diameter 1 mm. at the end and the tube is connected to an air tight bottle by a rubber tube and a bent glass tube fitting into the cork at the mouth of the bottle. The bottle is kept in connection

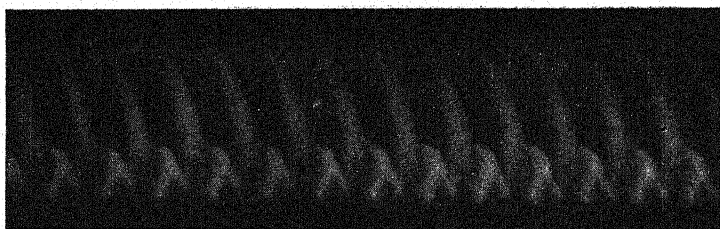


Fig.

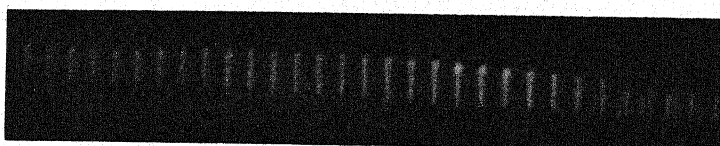


Fig.

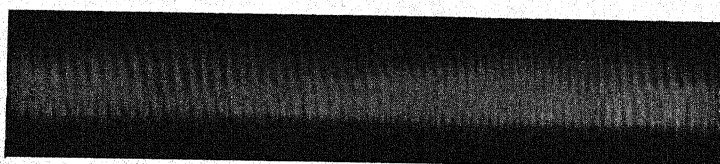


Fig.

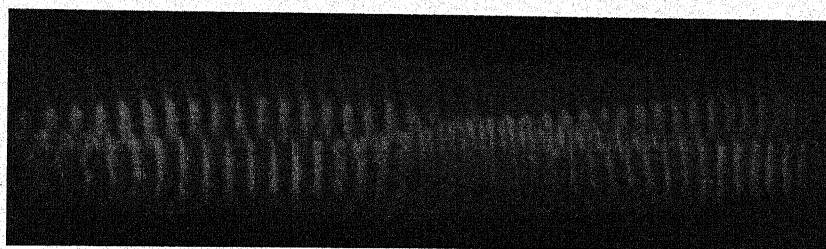


Fig.

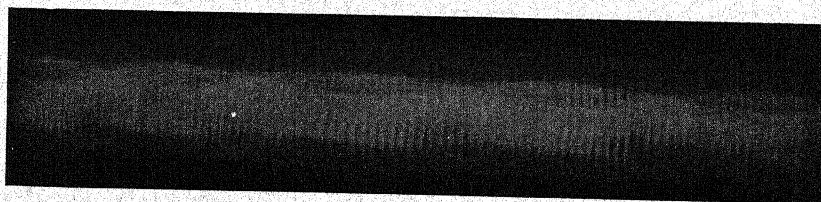


Fig.

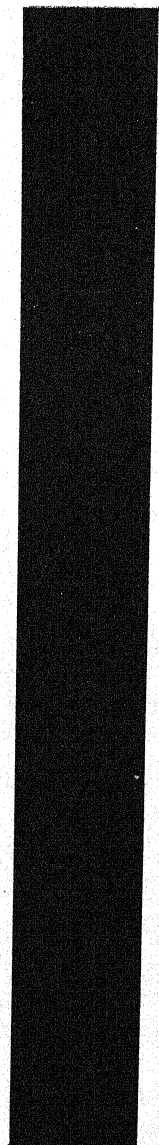


Fig. 6

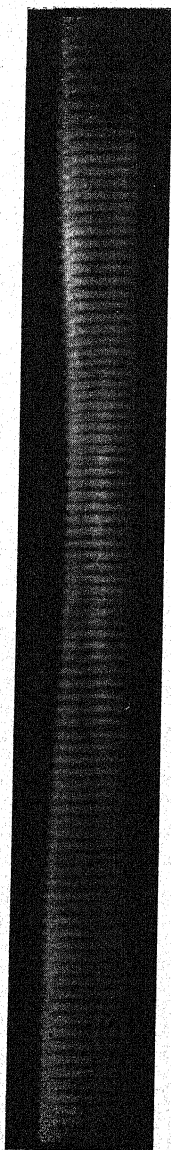


Fig. 7

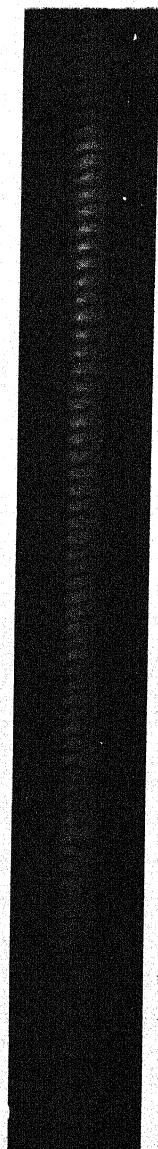
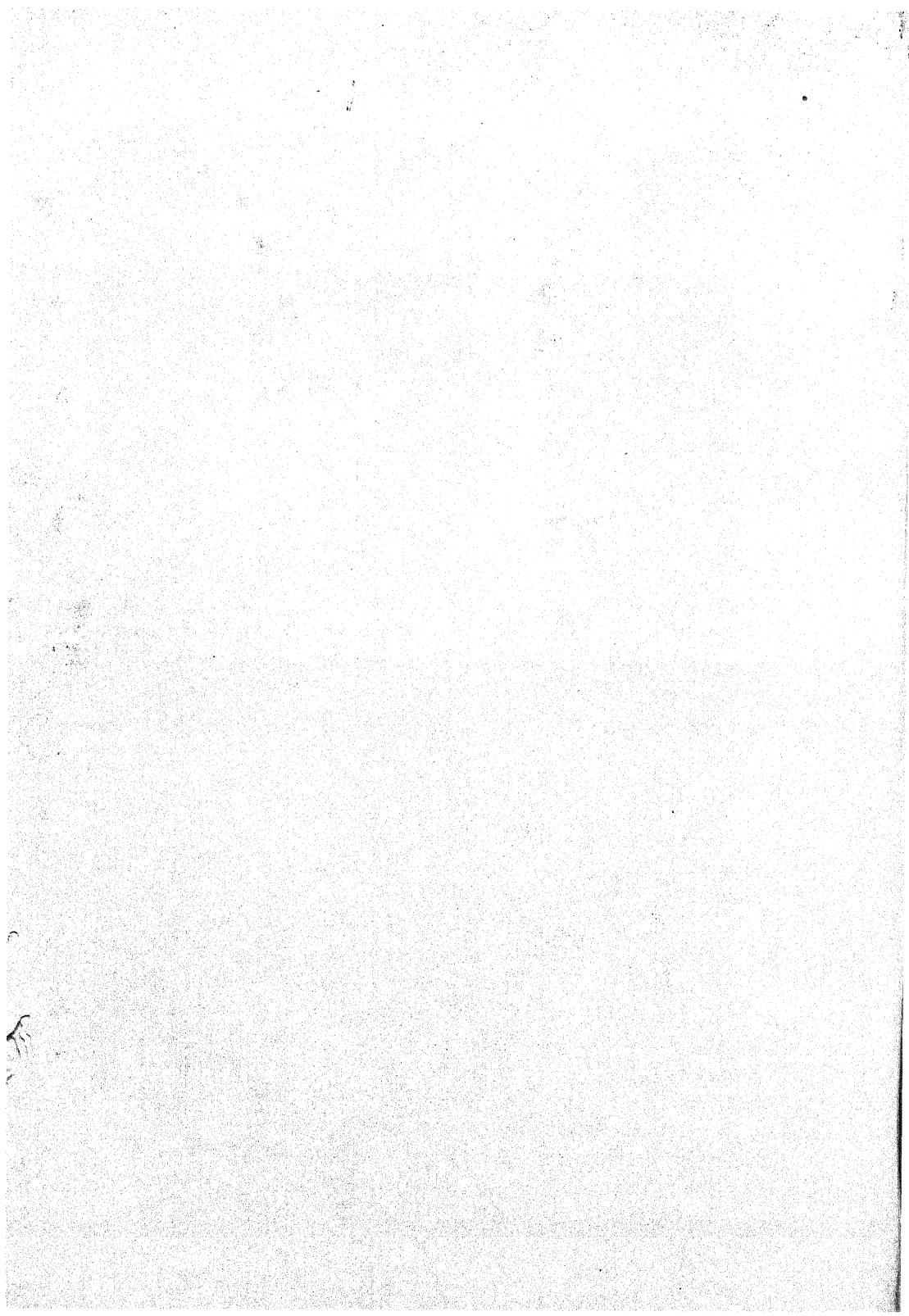


Fig. 8



with the gas tap, so that by adjusting the tap the amount of gas supplied to the flame can be regulated. The flame, being lighted, is introduced into the wider glass tube from below so that it is about $1/10$ of its way up the tube, when it begins to sing.

The Fundamental and the Octave in the Vibration.

On introducing the flame (which to begin with, may have a convenient length of about 2 cms.) into the tube, the flame takes up a symmetrical appearance and emits the fundamental of the tube nearly free from overtones at a height about $1/13$ of the tube. That the tone emitted here is the fundamental of the open tube on the assumed mode of stationary vibration with a node at the middle, has been verified with a sonometer tuned to a known pitch, e.g. it has been calculated that whereas the fundamental frequency for the particular tube used is 127, the pitch actually found is 125.

On reducing the flame slightly the higher harmonics are emitted along with the fundamental and can be heard with the unaided ear. This is shewn in the flame picture (Fig. 1, Plate II) in which the presence of the fundamental and the octave is clear. On very slightly reducing the flame length, the beating tone corresponding to the throbbing stage of the flame appears. This is described more fully below.

Reducing the gas supply still further, the flame for a moment becomes silent and then the octave alone bursts out with a symmetrical non-luminous flame, the corresponding pitch lying at about 254.

The Throbbing Stage.

In the throbbing stage of the flame, which comes in under conditions described above, the steady vibration of the flame is replaced by a throbbing with beats in which the audible and the visible effects keep exact time. The nature of the flame undergoes profound modification, for the single flame corresponding to the previous stage becomes double in some cases and triple in others, of which each component is in up and down motion relative to the others. On examining with a revolving mirror, cyclical variations in the amplitude of the different constituent parts of the flame are clearly observable, which are reproduced in Plates II. and III. Figs. 4 and 5 (Plate II)—shew the vibrations of a

long flame with three components, the middle component being in phase with the primary. The constant periodicity of the beats is very marked in Fig. 5 (Plate II).

Fig. 3, Plate II—shews the double flame, the beating octave being shewn by the smaller flame at the bottom, which waxes and wanes at epochs as shown also in Figs. 7 and 8 (Plate III). Fig. 8, Plate III, shews a small flame exhibiting the same effect; one cycle and halves of two following one after another. Fig. 2, Plate II and Fig. 6, Plate III, shew the nature of beats shewn by a small flame in which the octave is powerfully maintained.

The Overtones.

The fundamental tone given out at this stage is accompanied by a series of overtones of which the first four can be clearly heard. By suitable adjustment of the pressure and the length of the flame, higher harmonics of the motion of the flame and the tube can be brought into prominence, and the throbbing character of the flame can be made more complex than that indicated above, e.g., it has been noticed that with a flame 4 cms. long, the component with the smallest amplitude bends at its top towards the side of the tube, while it is exhibiting the cyclical variation in its amplitude. Again in the case of very small flames, exhibiting the effect, the top of the flame visibly quivers, while the lower part remains almost steady. It is worthy of remark here, that the beating tone is replaced by a steady flame without the cyclical variation of intensity, if the amount of gas supplied to the flame be slightly reduced or increased, the octave being strong in the former case and the fundamental in the latter.

Effect of position of flame in the tube.

The general effect of raising up the flame in the tube while singing is to bring in the higher harmonics, till the characteristic throbbing flame is obtained. On further raising, the flame goes out when it is about $1/7$ of its way up the tube. There is a lower limit of the position of the flame below which no singing can be obtained. This is about 9 cms. in the case of a tube 130 cms. long.

Analysis of the note by means of resonators.

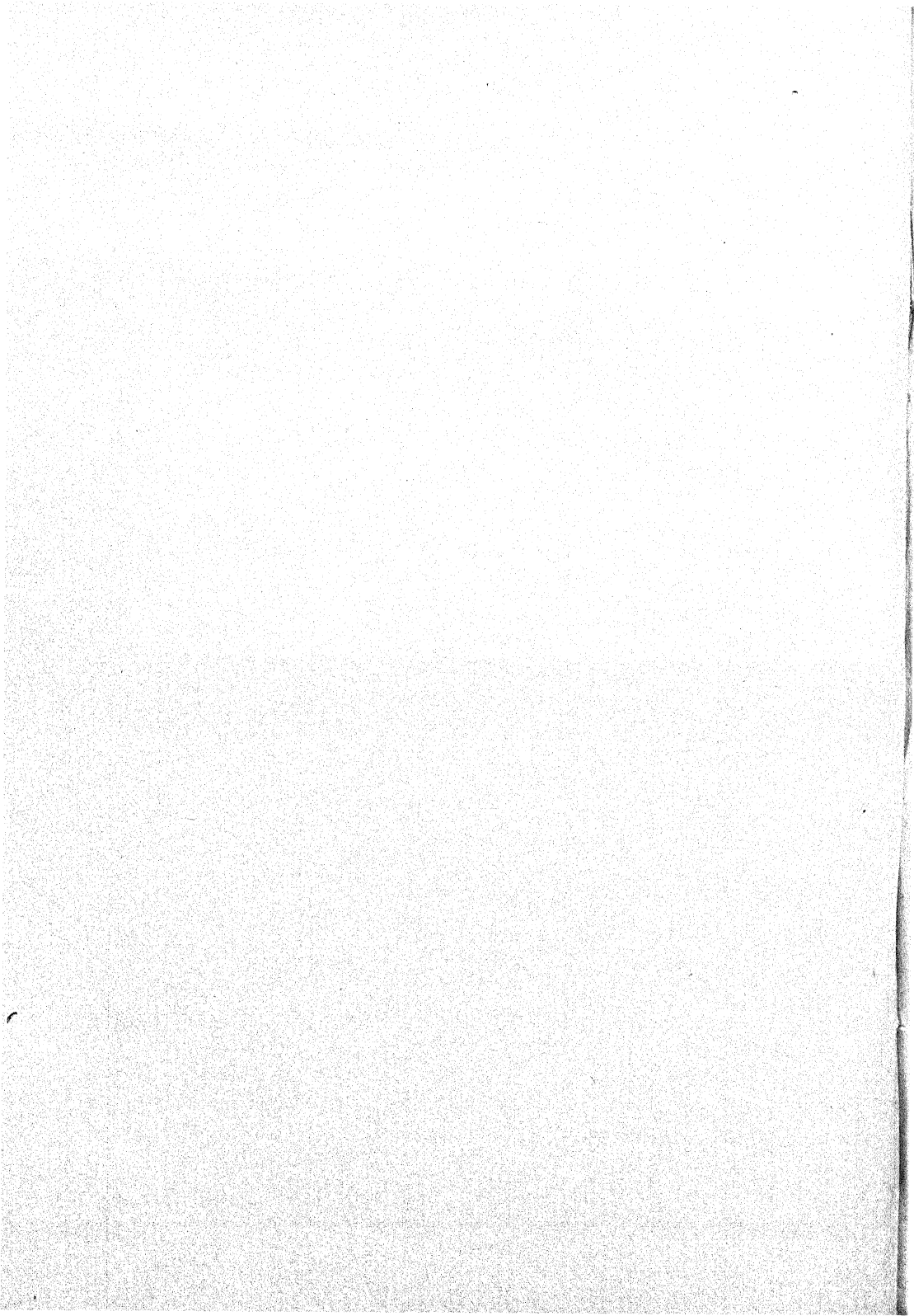
It is of interest to examine what particular tones in the sound given out by the throbbing flame are shewing the cyclical variation

attendant upon the sensation of beating produced. This point has been carefully examined in the case of the gas flame with the Helmholtz resonators, which shew that the principal fluctuations in the motion are in the amplitude of the second harmonic while the sound is of a highly compound character. The fundamental which is feeble is not fluctuating and so also is the third harmonic. The fourth harmonic is found to fluctuate with a periodicity of that of the second, which in one particular case was $1/6$ of a second.

III.—SUMMARY AND CONCLUSION.

Cyclical forms of vibration in the singing flame which are produced by adjustment of the flame length, so as to be intermediate between the stages when the fundamental on one side, and the octave of the motion on the other, are strongly brought out, have been experimentally studied and photographed. The visible effect of this form of vibration is the throbbing motion of the flame accompanied by beats in the octave. The phenomenon is determined mainly by the size of the flame (as determined by the quantity of gas supplied) and its position in the tube.

The experimental investigation was carried out in the University College of Science at the suggestion of Prof. C. V. Raman to whom the author wishes to express his best thanks for many valuable criticisms in the progress of the work.



VIII. Some Optical Notes.

By Dr. Phanindra Nath Ghosh, M.A., Ph.D., Rashbehari
Ghosh Professor of Applied Physics in the Calcutta
University.

(Plate IV.)

A. TRIPLE HORIZONTAL RAINBOWS ON A SOOTY WATER SURFACE.

The display of colours in the sky after a shower of rain must have attracted the notice of man probably from the earliest times. The circular shape of the coloured arcs and their similarity to the bows used by primitive people probably suggested the term rainbow. Whenever the sun shines on fine drizzling shower of rain, on the sprays from a waterfall, from a fountain, or even that of the fine misty cloud of water drops from the street house of the town when it waters the streets, the phenomena of the rainbow are seen. Some times one bow is seen but usually a slightly fainter one appears above the first bow. The colour sequence of the first bow is red outside and violet inside, while that of the other is that of just the opposite order, viz. violet in this case is the outer and red inner. The bows are arcs of concentric circles, the centre lying on the line joining the eye of the observer and the sun. Naturally no two people see the same bow, nay, neither do the two eyes of the same individual see the same bow.

There is a form of rainbow which is perhaps not so commonly seen. These are horizontal rainbows of which an admirable example was seen by me recently at Calcutta. After a calm cold winter night, the morning being clear, on one of the tanks in the Calcutta 'Maidan' (the tank just in front of Lindsay Street), I noticed a series of triple rainbows on the surface of the

water. They appeared to be of hyperbolic form, the bows lying successively within each other. The two inner ones had the same colour sequence, that is red without and violet within, while the third which is the faintest of the three, had the opposite colour sequence. All three of them lay along the same axis.

The remarkable feature about the bows which was noticed was that as the sun gradually rose, the positions of the bows were continually changing, the innermost one gradually receding further and further away from the middle bow. Further, the further arms of the hyperbolae got gradually closer and closer till after some time it was difficult to decide whether these figures were parabolae or hyperbolae.

On searching through the literature of the subject I found, the mention of a double horizontal bow of similar form by J. Dechant in the Proc. of Vienna Academy for June 1910. The geometry of the form of the primary horizontal bow has been worked out by Otobe in the Tokyo Proceedings, 1917. I shall endeavour here to explain the form of the other two bows and the other phenomena which appear along with it. Now regarding the cause of the phenomenon, I noticed that glistening dew drops were floating on the surface of the water, most probably on an oily and sooty layer on the surface of the water. It is the individual particles of soot which apparently served as condensation nuclei for the dew as the temperature gradually diminished during the night. I have tried to notice the phenomenon in other tanks in the maidan but though every one of them showed it to a certain extent, none showed it so well as the tank I have mentioned just now. My colleague and friend Mr. Satyendra Nath Bose also saw just a coloured band of light on the College Square tank, probably a portion of one arm of a hyperbola. The portion of the water surface lying between the primary and the secondary was remarkably dark and on careful observation, 3 to 4 bands of varying intensity could be seen inside the primary bow. These are the so-called supernumerary bows, explained by the theory of Airy.

Regarding the innermost bow, having the colour sequence same as that of the primary, it is to be remarked that it is much brighter than the secondary but less bright than the primary.

The geometrical theory of the rainbow is very well known as

due to the twice refracted and internally reflected beam of light from the water drops.

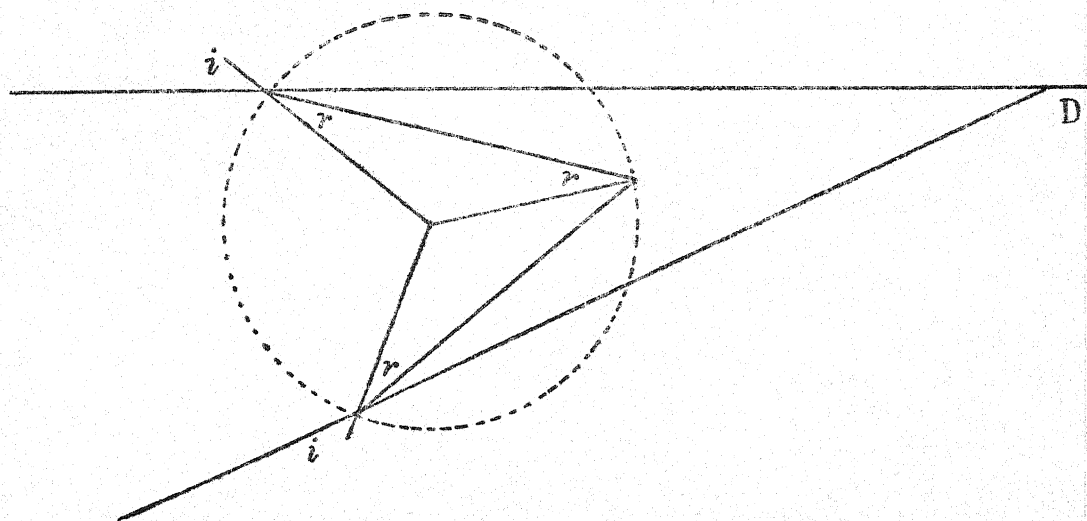


FIG. 1.

$$D = 2(i - r) + \pi - 2r$$

$$D = 2(i - r) + n(\pi - 2r)$$

The angle of incidence of the rays which suffer minimum deviation is given by

$$\cos i = \sqrt{\frac{\mu^2 + 1}{n^2 + 2n}}$$

taking $\mu = \frac{4}{3}$

n	Red	Violet
1	$\pi - 42^\circ.1$	$\pi - 40^\circ.22$
2	$\pi - 120^\circ.2$	$\pi - 125^\circ.48$

hence the primary bow marks an angle of

Violet	Red
41°	43°

and the secondary bow

Red	Violet
51°	54°

with the direction of the source. The innermost bow, as has been noticed by J. Dechant is due to the rays of sun getting inside the

$$\left(x + h \tan \theta + \frac{\gamma^2}{S}\right)^2 = (h^2 + \gamma^2) \left(\sec^2 \theta + \frac{2x}{S} + \frac{\gamma}{S^2}\right) \cos^2 \delta$$

making $S = -\infty$

$$(x + h \tan \theta)^2 = (h^2 + \gamma^2) \sec^2 \theta \cos^2 \delta$$

or

$$(x^2 \cos^2 \theta - \cos^2 \delta) + 2xh \sin \delta \cos \delta - \gamma^2 \cos^2 \delta + h^2 (\sin^2 \delta - \cos^2 \delta) = 0$$

When $\theta = \delta$, that is when sun's altitude reaches the value of 42° or 43° , we have

$$\gamma^2 \cos^2 \theta = 2xh \sin \theta \cos \theta + h^2 (\sin \theta - \cos^2 \theta)$$

$$\gamma^2 = 2h \tan \theta (x - h \cot 2\theta)$$

which is the equation of a parabola, of which the latus rectum $= 2h \tan \theta$, cutting the axis of γ at $h \cot 2\theta$.

For smaller values of θ , we have transferring the origin to the centre

$$y = 0$$

$$x = \frac{h \sin \theta \cos \theta}{\cos^2 \theta - \cos^2 \delta}$$

$$(\cos^2 \theta - \cos^2 \delta) x^2 - \cos^2 \delta y^2 - \frac{h^2 \sin^2 \delta \cos^2 \delta}{\cos^2 \theta - \cos^2 \delta} = 0$$

or

$$\frac{x^2}{\left(\frac{h \sin \delta \cos \delta}{\cos^2 \delta - \cos^2 \delta}\right)} - \frac{y^2}{\frac{h^2 \sin^2 \delta}{\cos^2 \theta - \cos^2 \delta}} = 1$$

which is the equation of a hyperbola.

B. FABRY-PEROT RINGS IN QUARTZ.

(Plate IV.)

The form of the Haidinger's Rings in mica has been recently carefully investigated by Chinmayandam,¹ who has shown that there are two sets of rings of approximately elliptic form, the superposition of which results in a fluctuation of the visibility of the rings seen in the field of view. In the case of mica which is a biaxial crystal of which the faces are practically perpendicular to the bisectrix between the optic axes, Chinmayandam has shown that the form of the lines of minimum visibility is identical with the 'isochromatic lines' in polarised light for a

¹ Proc. Roy. Soc. Ser. A, Vol. 95, January, 1919.

plate of twice the thickness. A similar result should be true for any crystal cut symmetrically with respect to the optic axis.

As an additional illustration of the theory of Haidinger's rings in crystals, the writer has recently studied the form of the Haidinger's rings in parallel plates of quartz. Three plates (each 5 mm. thick) accurately figured by Messrs. Adam Hilger were secured of which one was cut perpendicular to the optic axis, one parallel to it and one at an angle of 45° . For the present, observations have been made with the first two plates only. The Haidinger's rings as seen by the reflected light of a Cooper-Hewitt mercury-vapour lamp in the plates cut parallel and perpendicular to the axis respectively showed a remarkable difference. In the latter case, the rings were very clearly seen, in the former case, they were in general very confused. Further, when the usual arrangement of holding a glass plate at an angle of 45° above the crystal for observing the rings by reflection was adopted, it was found that in the case of the quartz cut perpendicular to the axis, rotating it in its own plane produced no effect on the rings, whereas in the case of that cut parallel to the axis, the rings were very clearly seen in four positions of the crystal and were extremely confused in the four intermediate positions. The reason for this becomes evident on remembering that the light is polarised by reflexion at the glass plate held at 45° . It was found in fact, that the angular positions of the rings were different when the optic axis lies in and at right angles respectively to the plane of incidence of the glass plate. In these positions, the rings were clearest, and in the intermediate positions, the confusion was a maximum.

The plates were then silvered on both sides and the Haidinger's rings observed by directly transmitted light. In this way, sharp rings similar to those observed with a Fabry-Perot etalon may be observed. The satellites of the green mercury lines could be easily seen, and there was a great difference in the phenomena observed with glass and quartz mercury vapour lamps respectively.

With the quartz plate cut perpendicular to the axis, only one set of rings was seen near the centre of the field, but at a greater angular distance from the centre of the field the bifurcation of the rings due to the double refraction in the quartz could be distinguished. It is, of course, well-known that quartz is an

Fig. 3

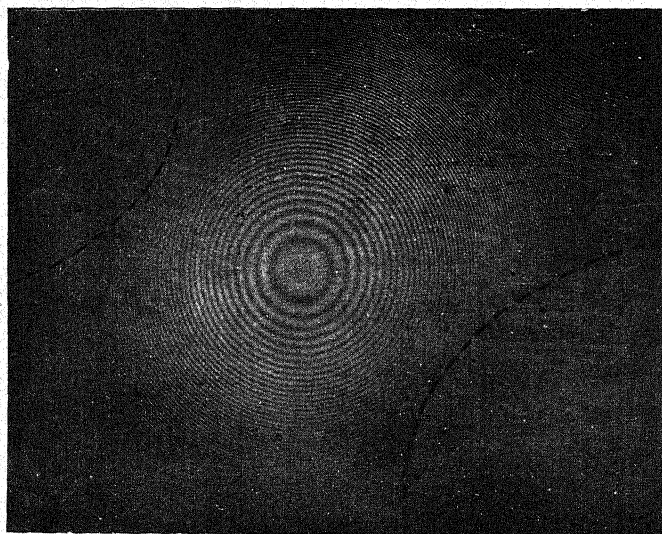
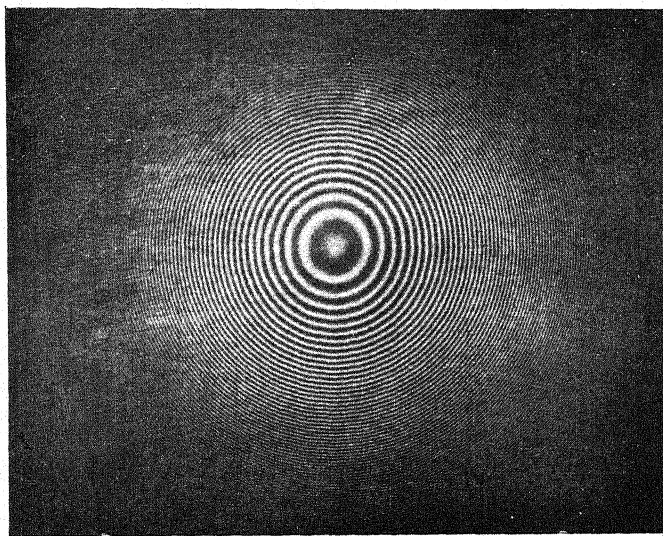


Fig. 4

Fabry Perot Rings in Quartz

optically active crystal and that a plane-polarised ray entering along the axis divides up into two circularly polarised rays travelling with two different velocities. It may therefore seem at first sight be surprising that even very near the direction of the optic axis, the rings do not show a bifurcation. It must be remembered however that one of the interfering rays has to pass twice through the crystal in opposite directions and hence the emergent ray is polarised in the same way as the incident light, and we get only one set of rings.

With the silvered quartz-plate cut parallel to the axis, two sets of rings are seen throughout the field. They are widest apart along a system of hyperbolic arcs which are evidently identical with the form of the isochromatic lines in polarised light of a plate of twice the thickness. The rings are therefore least clearly seen along these hyperbolic arcs.

Photographs of the Fabry-Perot rings in the silvered quartz-plates are reproduced as Figs. 3 and 4 in Plate IV. The form of the hyperbolic arcs of minimum visibility of the rings is indicated in Fig. 4. The great difference of the clearness of the rings in the two plates, due to their doubling in the latter can be clearly seen from the reproductions.

Further investigation of the subject, especially with the obliquely cut plate appears desirable. In this connection reference may be made to a paper on the Quartz Lummer-Gehrcke Plate by Takamine in the Proceedings of the Tokyo Physico-Mathematical Society, Vol. 8, page, 296, 1915.



Proceedings of the Indian Association for the Cultivation of Science.

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IX. On Quetelet's Rings in Mica.

By Nihal Karan Sethi, D.Sc., Professor of Physics and C. M. Soganí, M.Sc., Asst. Professor, Benares Hindu University.

(Plates V and VI.)

CONTENTS.

- I.—Introduction.
- II.—Observations in Ordinary Light.
- III.—Explanation of the Phenomena.
- IV.—Observations in Polarised Light.
- V.—Summary and Conclusion.

I.—INTRODUCTION.

When a distant source of white light is viewed by reflection at a sheet of mica of which the front surface is covered by a thin translucent film, the image of the source appears surrounded by gorgeously-coloured rings, the intensity of which is greatly enhanced by silvering the rear surface of the mica. The phenomenon is of the same nature as the well-known diffusion rings or Quetelet's rings exhibited by a dimmed mirror, but the use of the mica for the experiment gives rise, as has been pointed out in a recent paper¹ by Prof. C. V. Raman and Mr. G. L. Datta, to certain very interesting effects. If a fairly thin sheet of mica be employed, the rings are very broad and at the same time well-defined, and this makes it possible to easily observe them even when the light is incident on the sheet at large obliquities, and thus successfully to verify an effect which Stokes² had predicted from theory, but was himself unable to notice, namely that the Quetelet's rings observed at increasing angles of emergence at first become narrower and narrower, but widen again after reaching a minimum width and

¹ Phil Mag. XLII, November 1921.

² Mathematical and Physical papers, Vol. III, pp. 155-196.

become very broad at grazing emergence. But apart from this, the specially interesting feature of the Quetelet's rings in mica is their composite character which arises from the doubly-refracting property of the plate. The coloured rings are not uniformly visible over the whole field of observation, but show regions of maximum visibility separated by curved loci of minimum visibility along which the rings appear discontinuous or dislocated, the bright rings on one side of a line of minimum visibility appearing to run into dark rings on the other side and vice-versa. The phenomenon is somewhat analogous to that described in regard to Haidinger's rings in mica by Chinmayanandam¹ but differs from it in that in the present case it is not necessary to use highly monochromatic light to observe the phenomenon and further that it varies as the angle of incidence of the light is altered. The present investigation was undertaken at the suggestion of Prof. Raman in order to study the phenomenon more closely and to put forward a theoretical explanation.

II.—OBSERVATIONS IN ORDINARY LIGHT.

Though Quetelet's rings can be seen with white light, it is of advantage to use approximately monochromatic light so that the field over which the rings are distinct may be as large as possible. The observations described in the present paper were made with sunlight, the region of spectrum made use of being, however, considerably restricted with the aid of a monochromator. The mica employed was of the muscovite variety with an apparent angle of about 70° between the optic axes, and the back surface of this was heavily silvered. This resulted in greatly enhanced brilliancy of the rings and made it possible to observe them at much larger angles of scattering. For the scattering film, a thin film of ammonium chloride deposited by volatilization has been used throughout.

Under these conditions, by working in a darkened room and properly shielding the eye from extraneous light, it is possible to observe as many as 70 or 80 rings, and the lines of minimum visibility can be seen even at normal incidence. The appearance of the lines in this case is similar to that of Haidinger's rings and is indicated in Fig. 5.

¹ Proceedings of the Royal Society. A, Vol. 95, 1918.

The rings themselves are nearly circular or elliptical and the lines of minimum visibility possess approximately the shape of hyperbolae. There are also, as in Haidinger's rings two series of these curves with their axes nearly at right angles to each other and respectively parallel and perpendicular to the plane containing the optic axes. These are fixed relatively to the mica and rotate unchanged in position with respect to the centre of the ring system as the crystal is rotated in its own plane. And as in Haidinger's rings, the general form of these curves also does not vary much with the wave-length of the light or the thickness of the crystal, but their actual dimensions and position depend upon both. It may be noted, however, that the positions of these curves are not

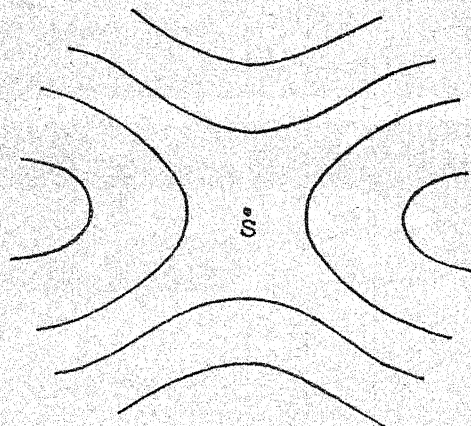


FIG. 5.

always indential with those of the corresponding curves in Haidinger's rings observed in the same plate with light of the same wave-length.

But when the light is made to fall obliquely on the plate and the reflected image of the source does not coincide with the centre of the ring system, the above-mentioned fixity of the curves with respect to the mica disappears, and as the mica is rotated not only do they change in position but they also appear to grow or diminish in size. Yet their axes always remain fixed in the same directions as in the case of normal incidence.

The most interesting and striking facts are, however, revealed, when the incidence is such that the regularly-refracted rays nearly

coincide with the direction of an optic axis. When this coincidence is exact, the curves of dislocation no longer all have the form of hyperbolae but some of them are closed curves surrounding the direction of the optic axis, and the reflected image of the source. Fig. 1 in Plate V corresponds to this position. Though it does not include the entire field of view as seen with the eye, two complete closed curves will be seen there, together with a third curve which is not closed. It will be noticed that these strongly resemble the isochromatic curves seen in biaxial crystals in convergent polarised light.

If now, the direction of incidence is gradually altered, as for example by slowly rotating the mica in its own plane these curves appear to increase in size and alter their form till they assume

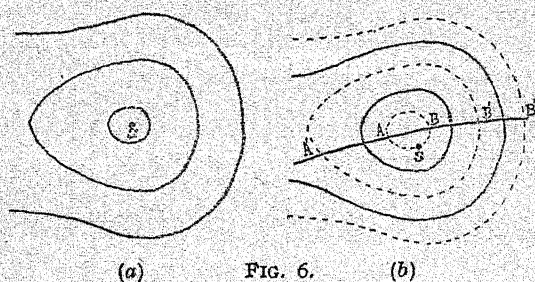


FIG. 6.

the curious appearance shown in Fig. 2 in Plate V, and represented in Fig. 6 (b), by full line. Fig. 6 (a) represents the previous case.

We still find closed curves but they are now crossed by an additional line of dislocation, more or less straight in form, and the visibility of the rings is bad in considerable regions round the crossings. On a further change in the direction of incidence the original form of closed curves is restored, but the reflected image of the source, *S*, is no longer at their centre. The same appearances repeat periodically as the change in the direction of incidence is continued.

Fig. 3 in Plate VI shows the curves obtained when the plane of incidence is nearly perpendicular to the plane containing the optic axes.

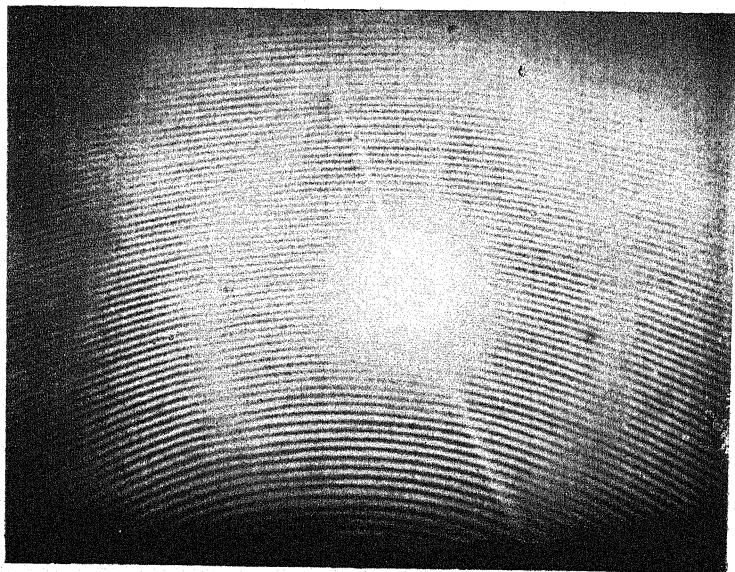


FIG. 1.

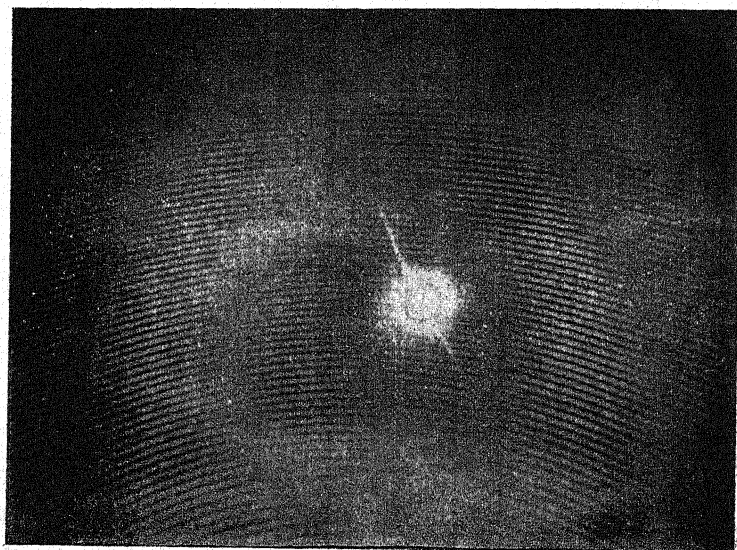


FIG. 2.

III.—EXPLANATION OF THE PHENOMENA.

According to the theory put forward by Stokes,¹ Lommel² and Exner,³ the interferences in Quetelet's rings in the case of isotropic media like glass and air are due to the two sets of rays diffracted at the dimmed surface, in one case at entry and in the other case at emergence, namely the scattered-transmitted and the transmitted-scattered rays. In this simple case only a single system of circular rings is obtained.

In the case of doubly-refracting substances like mica, however, instead of a single transmitted-scattered and a scattered-transmitted ray, there will correspond to a given direction of scattering two rays of each set as represented diagrammatically in Fig. 7.

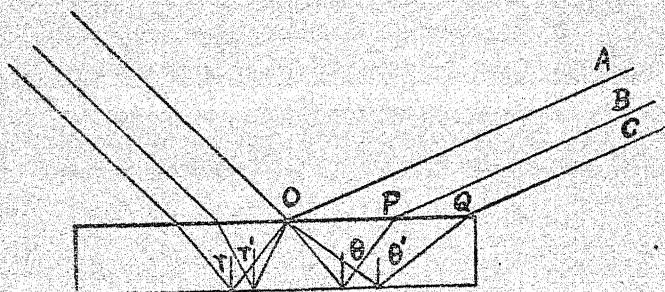


FIG. 7.

In this figure, there are two transmitted-scattered rays along OA; and PB and QC are the two scattered-transmitted rays. And each of these is polarised. According to the well-known Fresnel-Arago laws of interference of polarised light, interference between two or more rays of polarised light is possible only when all these rays are derived from the same polarised beam. Let us therefore resolve the incident unpolarised beam into two components polarised in the principal directions. Each one of these components will give rise to rays capable of interfering with each other but not with the rays derived from the other component. We should therefore consider them separately.

Taking first one of these components only we get a single

¹ *Loc. cit.*

² *Ann. der Phys.* Vol. VIII, p. 193 (1879).

³ *Sitzungsberichte of the Vienna Academy*, XC, p. 827 (1884) and *Ann. der Phys.* Vol. IX, p. 239 (1880).

transmitted-scattered ray along OA together with scattered-transmitted rays both along PB and QC . The vibrations in the rays PB and QC are perpendicular to each other, but neither of these is in general parallel to the vibration in OA , which may, however, be further resolved into two perpendicular components parallel respectively to the vibrations in PB and QC . One component will then interfere with PB and the other with QC , giving us the two systems of rings

$$(a) \delta_1 = 2t (\mu_0 \cos r - \mu \cos \theta)$$

and

$$(b) \delta_2 = 2t (\mu_0 \cos r - \mu' \cos \theta')$$

where t is the thickness of the mica, r , θ and θ' are the angles indicated in Fig. 7, and μ_0 , μ and μ' are the refractive indices along the directions corresponding to the rays OA , PB and QC respectively. As the acute bisectrix of the angle between the optic axes is very nearly normal to the surface of mica, the refractive index is the same after reflection as before it. These systems of rings, it may be noted, are polarised in perpendicular planes and consequently are independent of each other.

The other component of the incident light will similarly give rise to two more systems of rings

$$(c) \delta_3 = 2t (\mu_0' \cos r' - \mu \cos \theta)$$

and

$$(d) \delta_4 = 2t (\mu_0' \cos r' - \mu' \cos \theta')$$

The rays which produce these rings (c) and (d) being incapable of interfering with those which produce (a) and (b), we have now not one system of rings as in isotropic media, not even two as in Haidinger's rings in a doubly-refracting crystal but actually *four* absolutely independent systems of rings.

Following Chinmayanandam's method, it can be easily shown that when θ and θ' are not large, these rings are all elliptical in form, the equation of n th order rings being respectively

$$(a) \quad c^2 x^2 + a^2 y^2 = k \left\{ v_0^2 - \frac{a^2}{4t^2} (n\lambda - 2t\mu_0 \cos r)^2 \right\}$$

$$(b) \quad b^2 x^2 + c^2 y^2 = k \left\{ v_0^2 - \frac{b^2}{4t^2} (n\lambda - 2t\mu_0 \cos r)^2 \right\}$$

$$(c) \quad c^2 x^2 + a^2 y^2 = k \left\{ v_0^2 - \frac{a^2}{4t^2} (n\lambda - 2t\mu_0' \cos r')^2 \right\}$$

$$(d) \quad b^2 x^2 + c^2 y^2 = k \left\{ v_0^2 - \frac{b^2}{4t^2} (n\lambda - 2t\mu_0' \cos r')^2 \right\}$$

where a , b , c , are the principal velocities in the mica, v_0 is the velocity in air and h is a constant, depending on the distance of the eye from the mica. It will be seen that the rings (a) and (c) are similar ellipses with their major axis lying in the plane of the optic axes, while rings (b) and (d) are also similar ellipses but with their major axis in the perpendicular direction.

The visibility of these rings will be minimum wherever the maxima due to any one system coincide with the minima due to any other. There are thus six possible curves of minimum visibility given by the following equations:—

$$\delta_1 - \delta_2 = 2t(\mu' \cos \theta' - \mu \cos \theta) = (2n + 1) \frac{\lambda}{2} \dots\dots\dots (1)$$

$$\delta_3 - \delta_4 = 2t(\mu' \cos \theta' - \mu \cos \theta) = (2n + 1) \frac{\lambda}{2} \dots\dots\dots (2)$$

$$\delta_1 - \delta_4 = 2t \left\{ (\mu_0 \cos r - \mu'_0 \cos r') - (\mu \cos \theta - \mu' \cos \theta') \right\} = (2n + 1) \frac{\lambda}{2} \dots\dots (3)$$

$$\delta_2 - \delta_3 = 2t \left\{ (\mu_0 \cos r - \mu'_0 \cos r') + (\mu \cos \theta - \mu' \cos \theta') \right\} = (2n + 1) \frac{\lambda}{2} \dots\dots (4)$$

$$\delta_1 - \delta_3 = 2t(\mu_0 \cos r - \mu'_0 \cos r') = (2n + 1) \frac{\lambda}{2} \dots\dots\dots (5)$$

$$\delta_2 - \delta_4 = 2t(\mu_0 \cos r - \mu'_0 \cos r') = (2n + 1) \frac{\lambda}{2} \dots\dots\dots (6)$$

It will be easily seen, however, that as equations (5) and (6) do not involve θ or θ' they do not represent any curves of minimum visibility. They are due to rings which are similar ellipses and only signify that the rings (a) and (c) as well as (b) and (d) are out of step in the whole field for certain directions of incidence. This would have altogether spoiled the visibility of the rings throughout the field, but for the fact that the rings (a) and (c) as well as (b) and (d) are not, in general, equally intense as will be shown further on.

Curves (1) and (2) are identical and can easily be seen to be the isochromatic curves observed in convergent polarised light for a plate of thickness $2t$. These are also identical with the curves observed in Haidinger's rings in the same plate.

Equations (3) and (4) can be re-written as

$$\begin{aligned} 2t(\mu' \cos \theta' - \mu \cos \theta) &= (2n + 1) \frac{\lambda}{2} - 2t(\mu_0 \cos r - \mu'_0 \cos r') \\ &= (2n + 1) \frac{\lambda}{2} - k \quad \text{say} \end{aligned}$$

and

$$2t (\mu' \cos \theta' - \mu \cos \theta) = - \left\{ (2n+1) \frac{\lambda}{2} - k \right\}$$

which shows that curves (3) and (4) are also identical. These too are lemniscates whose poles are identical with those of the isochromatic curves referred to above, though, ordinarily, they differ in position from them.

To sum up, we are left in general with two sets of curves of minimum visibility of which the first is independent of the direction of incidence and is fixed relatively to the crystal and the second varies in position with the direction of the incident light on account of the constant 'k' involved in their equations. We might call these the 'fixed' and the 'variable' sets respectively. For directions of incidence for which $k=n\lambda$, the two systems coincide, while for the directions $k=(2n+1)\frac{\lambda}{2}$, the curves of one system fall midway between those of the other. In consequence of this, the two systems alternately get in step and out of step, as the direction of incidence is altered. In addition to these, under suitable conditions of intensity, there may also appear for certain directions of incidence, some lines of minimum visibility corresponding to equations (5) and (6).

It may be noticed that as in the direction corresponding to an optic axis, $\delta_1 - \delta_2 = \delta_3 - \delta_4 = 0$, the order of the fixed curves increases in all directions as we move away from the point, and the visibility here is always maximum so far as this system is concerned. In the case of the variable curves, however, the point where a similar condition is satisfied is the reflected image of the source and the visibility in its neighbourhood therefore always remains maximum.

It should not be supposed, however, that both these systems will be equally visible everywhere; for the four sets of rings giving rise to them are not of equal intensity all over the field. In the region round the reflected image of the source to which alone the observations have to be confined on account of the feebleness of scattering at large angles, the principal directions of vibration corresponding to θ and θ' do not, in general, differ much from those corresponding to r and r' of the regularly transmitted light. For this reason, of the two disturbances along PB and QC

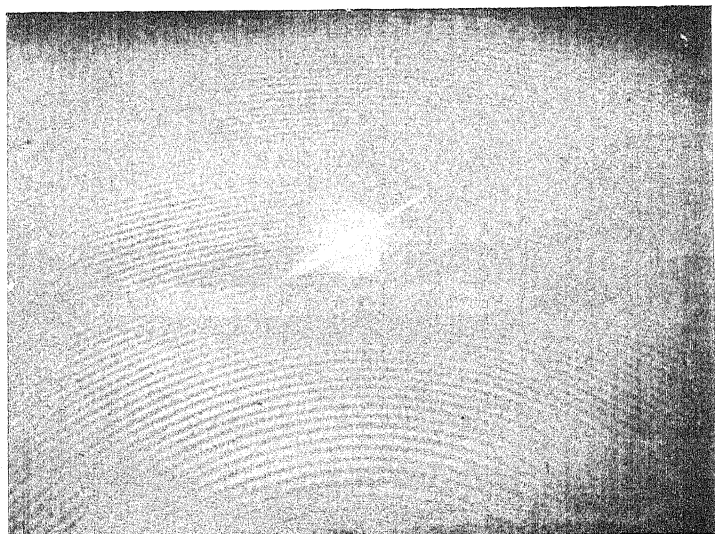


FIG. 3.

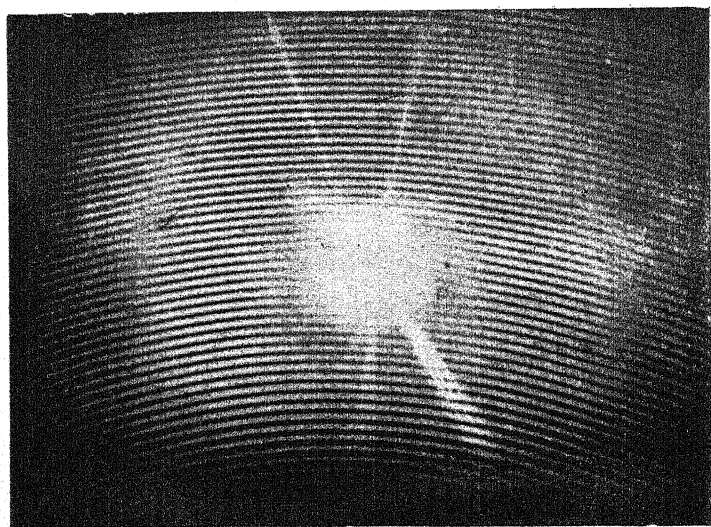
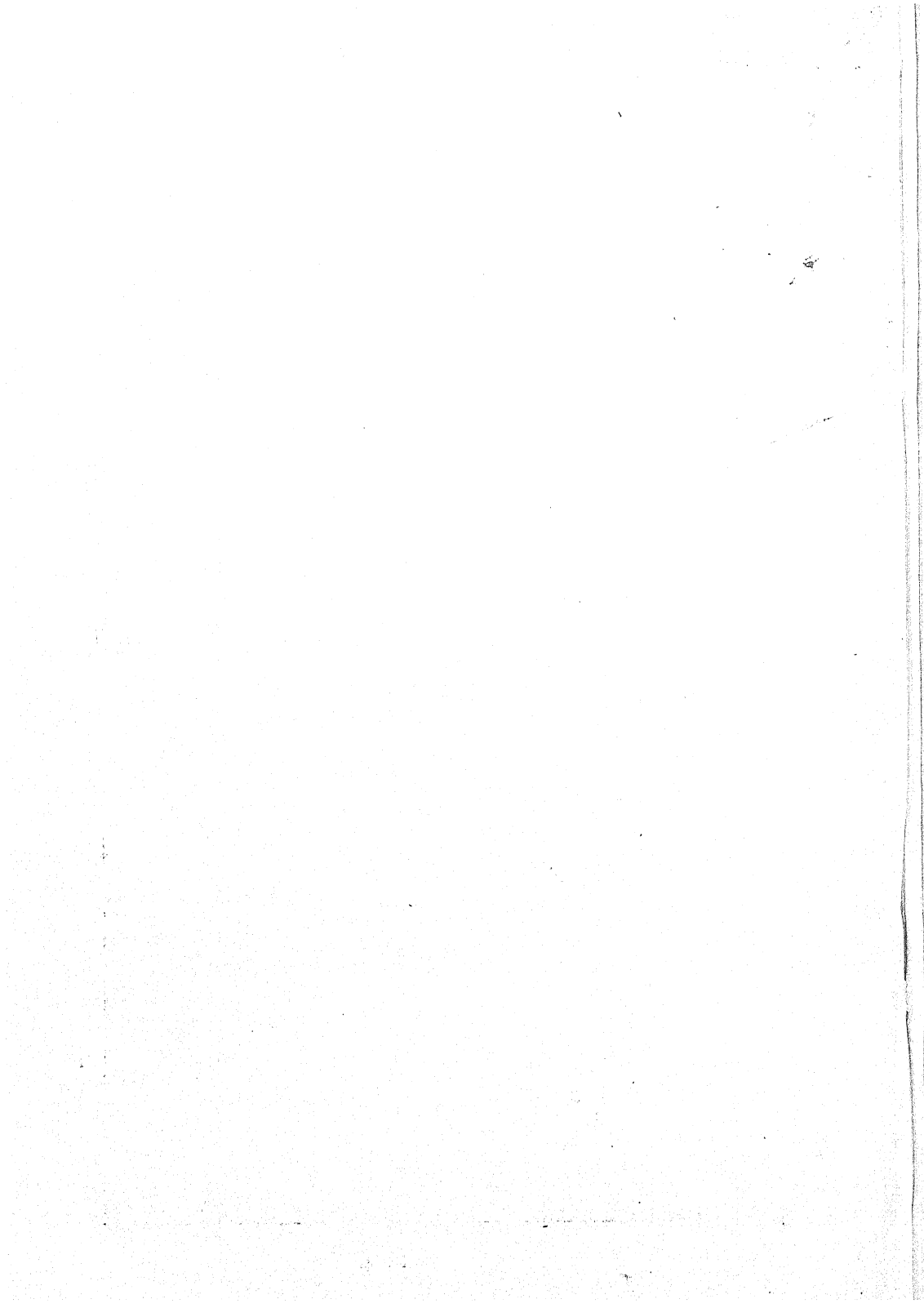


FIG. 4.



due to one of the two components of the incident light, only one will have an intensity comparable with that of OA , and therefore of the rings (a) and (b) only the ring (a), say, will be formed. Similarly of the rings (c) and (d) due to the other component, only one (d) will be visible along with (a). Thus of the two systems of curves of minimum visibility, the fixed one cannot be expected to be observed in this region. Only the variable system will be present.

If, however, the incidence nearly corresponds to the direction of an optic axis, even in the narrow region round the reflected image of the source, the principal directions of vibration change from point to point so that it is possible to find directions in which the intensities of the two rings (a) and (b) as also of (c) and (d) are comparable. Portions of the fixed curves ought to be observable in these directions in addition to the variable curves which are present everywhere, for wherever rings (a) and (d) are weak, rings (b) and (c) will become prominent giving rise to the same curve of discontinuity again. The presence of these portions of the fixed discontinuities will not, however, make any difference in the appearance of the variable curves for directions of incidence such that $k=n\lambda$, owing to the overlapping of the two systems (case illustrated in Fig. 1). But whenever $k=(2n+1)\frac{\lambda}{2}$, these will fall between the variable curves and spoil the visibility of the rings there. It is this case which is illustrated in Fig. 2. That here k was really equal to $(2n+1)\frac{\lambda}{2}$, was verified by polarising the incident light and observing the regularly transmitted light from behind the mica with a quarter-wave plate and a nicol. In connection with this, attention may also be drawn to a very curious line of minimum visibility seen running almost straight through the middle of the closed curves. This belongs neither to the fixed nor to the variable system, but is a part of the minimum visibility represented by equations (5) and (6) and makes its appearance because all along this line, which is an isogyre running through the portions of the fixed discontinuities visible in this case, the four rings (a), (b), (c) and (d) are more or less equally strong.

IV.—OBSERVATIONS IN POLARISED LIGHT.

(a) Incident Light Polarised.

In general, when the incident light is polarised no curves of minimum visibility are seen, but careful observation shows that in certain directions, there are small regions of dislocation extending barely over half a dozen rings. In the cases corresponding to Figs. 1 and 6 (a), two such portions at nearly opposite points of each curve are visible. These go on rotating round the curves as the incident plane of polarisation is rotated interchanging these positions after a rotation of the nicol through 90° and completing the round in 180° . In the special cases when the curves have the appearance shown in Fig. 2 and Fig. 6 (b) the behaviour is slightly different. When the incident plane of polarisation coincides with one of the principal directions, the discontinuities are found in the positions marked A and B in Fig. 6 (b). As the plane of polarisation is rotated, these discontinuities rotate a little and expand until at 45° nearly, the whole of the curve in Fig. 2 makes its appearance. On further rotation, the discontinuities contract again until at 90° we are left with the same original appearance.

(b) Incident Light Unpolarised and Rings observed through a nicol.

Observations in this case are similar to those in case (a).

(c) Incident Light polarised and the rings observed through a nicol.

For any one of the directions of incidence when the closed curves appear quite distinctly in unpolarised light, we have already seen that in every position of the polarising nicol, only two small portions of each of these curves are visible. On observing the rings through a second nicol the discontinuities are found in exactly the same positions for one particular direction of this nicol. On rotating it however, they are found to rotate round the curves interchanging their positions after 180° and not after 90° as in cases (a) and (b). At 90° , when the discontinuities occupy intermediate positions, some very interesting effects are observed. For there appears in the field of view, a dark line running along an *isogyre* through the discontinuities together with dark curves similar in form to the bright curves of minimum visibility seen in unpolarised light but falling midway between them. At the same time, the visibility of the rings becomes bad throughout the field, and but for the rings,

still discernible, the whole appearance strongly resembles that usually obtained in convergent polarised light.

For the intermediate directions of incidence which give us the complicated appearance in Fig. 2, the effects observed when the incident light is polarised in a principal plane are similar though differing in important respects. We get the same rotation of the discontinuities, but this, instead of taking place along the closed curves actually visible in unpolarised light, follows the invisible intermediate curves. As a consequence of this, one of the discontinuities passes through the image of the source itself, a very interesting fact indeed. We also get a dark line and the dark curves, but these again do not fall between the curves visible in ordinary light but coincide with them.

We will now consider the explanation of the foregoing effects confining ourselves to the two important cases when

$$(1) \ k=n\lambda \quad \text{and} \quad (2) \ k=(2n+1)\frac{\lambda}{2}.$$

(a) *Incident light polarised.*

In this case when, $k=n\lambda$, the transmitted-scattered ray is evidently plane-polarised. Its vibrations may, as before be resolved into two components parallel respectively to the vibrations along PB and QC , and we shall get two ring systems with equations identical with those of (a) and (b). In the directions of scattering in which the angle between the directions of vibration of PB and QC and of the transmitted-scattered ray OA is 0° or 90° , only one of these rings will be present. In directions in which the angle is about 45° , the two rings are more or less of equal intensity and those portions of the curves of minimum visibility which lie in these directions will therefore be visible. As the plane of polarisation of the incident light is rotated, the resultant direction of vibration along OA will be rotated also and this condition will be satisfied in some other directions of scattering with the result that these discontinuities will appear to rotate round the curve as seen in unpolarised light.

When, however, $k=(2n+1)\frac{\lambda}{2}$, although the transmitted-scattered ray OA is still plane-polarised, yet as the two perpendicular components to which it is due are in opposite phases, its phase is indeterminate and we cannot proceed as above. It is therefore necessary to consider both the components of OA separately.

This will again give us four ring systems as in the case of unpolarised light. Being polarised perpendicularly the rings (*a*) and (*b*) are independent as are also the rings (*c*) and (*d*). The rings (*a*) and (*c*) or (*b*) and (*d*) are similarly polarised and are therefore not strictly independent, but being out of step every where in the field, they may also be regarded as independent. The case is therefore identical with that of unpolarised light except that in this case the two components into which the incident light is divided are not in general equal in intensity. Thus, for example, when the incident light is polarised in one of the principal directions, only two rings say (*a*) and (*b*) will be formed. These will as before give rise to exactly the same small portions of the discontinuities which in the case of unpolarised light fall between the variable curves and spoil the visibility of the rings in that region. But when the incident light is polarised in a plane inclined at 45° to the principal directions, both the components will be equally strong and the case becomes identical in all respects with that of unpolarised light and all the lines of minimum visibility as seen in ordinary light will make their appearance. This is in complete agreement with the observations already recorded.

The effects observed when the rings are viewed through a nicol and the incident light is unpolarised can be easily explained on similar lines, if we remember that this nicol does not interfere with the actual formation of the rings but only cuts off a certain portion of the light from them and alters their relative intensities to different extents in different directions.

We will now proceed to explain the formation of the very interesting dark curves that appear when the rings are viewed through a nicol and the incident light is also polarised. We have already seen that in the case when $k=n\lambda$, the transmitted-scattered ray is polarised. The plane of polarisation of this ray is nearly the same as that of regularly transmitted-reflected ray all over the narrow field in its neighbourhood. It follows therefore that when the analysing nicol is crossed to this direction, the transmitted-scattered ray vanishes at every point, and we are left with the two perpendicularly polarised scattered-transmitted rays, the phase-difference between which is given by

$$\delta = 2t (\mu \cos \theta - \mu' \cos \theta')$$

It will be seen that this phase-difference is constant along curves identical with the isochromatic curves and is a whole number of wave lengths along curves

$$2t (\mu \cos \theta - \mu' \cos \theta') = n\lambda$$

These can be easily seen to lie between the curves of the fixed system of minimum visibility. All along these curves, the two scattered-transmitted vibrations will combine to give us a plane vibration, more or less coinciding in direction with the regularly transmitted vibration and will therefore vanish simultaneously with it. No light will thus reach us from these curves which will in consequence appear dark. The presence of the isogyric line in this case hardly needs an explanation. In the case when $k = (2n+1)\frac{\lambda}{2}$, although the transmitted-scattered ray is still polarised yet the above conditions are not satisfied except when the incident light is polarised in a principal plane. The result is that the dark curves appear fully formed in this case only.

V.—SUMMARY AND CONCLUSION.

1. The interesting effects due to double refraction observed in Quetelet's rings in mica by Prof. C. V. Raman and Mr. G. I. Datta have been studied in detail and a theoretical explanation has been put forward.

2. It has been shown that in this case we have *four* independent sets of Quetelet's rings nearly elliptical in form in place of the single set of circular rings in isotropic media, and the interactions of these give rise to two different systems of curves of minimum visibility.

3. One of these systems of curves is *fixed* relatively to the mica and is identical with that observed in Haidinger's rings in the same plate. It consists of isochromatic curves or lemniscates similar to those seen in convergent polarised light. Only small portions of these curves are, however, visible because all the four ring systems are not equally intense everywhere in the field.

4. The other system also consists of lemniscates with the same poles as the fixed system, but the exact size and positions of the curves of this system undergo considerable changes as the direction of incidence is altered, alternately coinciding and

getting out of step with the curves of the fixed system. These curves are fully visible everywhere.

5. When the two systems are out of step, an additional line of dislocation is also observed. This runs nearly along an isogyre and is observable in the neighbourhood of the optic axis. (Plates V and VI.)

6. When the incident light is polarised, or when the rings are observed through a nicol, the appearance is altogether different and in general only very small portions of the curves are visible in isolated localities. (Fig. 4 in Plate VI.) These change their positions and rotate round the curves when the polarising or the observing nicol is rotated.

7. But when the incident light is polarised and the observations are also taken through a nicol, we get certain dark curves and an isogyric line in place of the usual bright curves of minimum visibility, the whole appearance strongly resembling that observed with crossed nicols in convergent light. Even these special effects in polarised light have been shown to follow easily from the theory advanced in this paper.

In conclusion, the writers wish to express their cordial thanks to Prof. C. V. Raman for the suggestion of the problem as well as for the interest he has taken in the work.

*Physical Lab., B.H.U.,
Benares,
The 2nd May, 1922.*

X. On the Colours of Tempered Steel and other Tarnished Metal Surfaces.

By Brojendra Nath Chuckerbutti, D.Sc., Assistant Professor
of Physics, Calcutta University.

(Plate VII.)

CONTENTS.

- I.—Introduction.
- II.—Experimental details.
 - (i) Preparation of metal plates.
 - (ii) Microscopic observation of the plates.
- III.—Description of phenomena observed.
 - (i) Colour and polarisation of the reflected light.
 - (ii) Colour and polarisation of the scattered light.
 - (iii) Quantitative study of the scattered light.
 - (iv) Study of the reflected light with a Babinet compensator.
- IV.—Physical theory of scattered colours.
- V.—Theoretical results for copper and iron plates.
- VI.—Discussion of the results.
- VII.—Summary and Conclusion.
- VIII.—Explanation of the Photographs.

I.—INTRODUCTION.

The well-known and characteristic tints that appear on a metal surface when it is heated in air, have been the subject of interest for a long time. When, for instance, a polished steel plate is heated over a Bunsen flame, rings of colour appear forming a periodic succession, somewhat in the manner of Newton's rings, and the explanation usually put forward is that they are due to the interference of the light reflected at the surfaces of a thin film formed upon the metal as the result of oxidation. In a recent paper, Mallock¹ has questioned the correctness of this familiar

¹ Mallock—Proc. Roy. Soc. Vol. XCIV, Ser. A, August, 1918. See also Nature December 1, 1921.

explanation. On trying to polish gently a tempered steel surface, he found that there was no change in the colours observed as the film was gradually removed. Thus, for instance, a blue-tempered steel remained blue throughout the process of polishing until the clear metallic surface was reached, although there was a visible change in the intensity of the reflected light.

Observing the reflected colours through a nicol oriented so as to quench the reflected light as far as possible, Mallock found that as the angle of incidence is increased beyond that required for maximum polarisation, the colours changed with obliquity almost in the same manner as in case of high order Newton's rings. But the thickness of the film necessary for the production of such high order rings, is far greater than that of the actual thickness of the film upon the metal surface. As the results of these observations, he concluded that "the colours must be due to some form of selective opacity depending on damped molecular periods comparable with the wave-period rather than on a structure comparable with the wave-length."

Sir George Beilby¹ has also made observations on the subject of these coloured films. According to him the film is an aggregate in open formation, through which oxygen molecules can penetrate to the metallic surface underneath. The colour of the film formed depends not only on the temperature to which the metal is raised but also upon the time for which the heating is continued.

Recently, in a note published in *Nature*, Prof. C. V. Raman² has put forward a different explanation of these colours. He drew attention to the fact that the plates exhibit a colour by scattered light which is complementary to that shown by reflected light. As the result of his observations on the colour and polarisation of the light reflected and scattered from the surfaces, Raman has put forward the view that the colours under discussion are in the nature of *diffraction effects* arising from a film which is not continuous, but has a close-grained structure.

Now, the theory that the colours are intrinsic put forward by Mallock and apparently supported by Beilby, presents serious difficulties and cannot be accepted. Apart from all other consi-

¹ Beilby—"Aggregation and Flow of Solids" (Sections 3 & 10).

² Raman—*Nature*, January 26, 1922.

derations, the very fact that the colours on the plate form an approximately periodic sequence, strongly suggests that they are in great part due to either interference or to diffraction, and only in a minor degree to *intrinsic* colour which cannot, according to any reasonable supposition be expected to exhibit a periodic sequence with the growth of the film. Since the observations of Mallock appear to exclude simple interference as the origin of the phenomena, we are left with the only other alternative left as an explanation, that is, that the colours arise from the scattering or diffraction of light by a granular structure in the manner suggested by Raman.

The present author took up this investigation with the object of making a detailed study of the whole phenomenon and to collect experimental evidence for a comprehensive explanation of the facts. The principal task was to obtain a set of highly polished metal plates. For in case of plates not properly polished, the colour sequence, especially of the scattered light as distinguished from that regularly reflected cannot be satisfactorily observed, and it is not possible to derive useful information with their aid. After prolonged trial, however, the necessary technique was acquired, and the problem has been thoroughly studied from various stand points, and also with plates of different metals. As has been observed by Raman,¹ copper (which, however allows a very high polish easily) shows the phenomena very beautifully and is found rather the most suitable for the different experiments.

Now, Beilby² has shown that the process of polishing a metal surface causes the surface layer to "flow" as a liquid, and thus the polished surface is that of the metal in the amorphous state, that is to say, the metal molecules are distributed at random. It is not unreasonable to suppose that the subsequent process of heating results in the formation of aggregates of these molecules, some of which are altered by oxidation. Hence the problem reduces itself to the determination of the scattering of light by

¹ Raman—*loc. cit.*

² Beilby—Hunter memorial lectures, Glasgow, 1903, page 46, also "Aggregation and Flow of Solids."

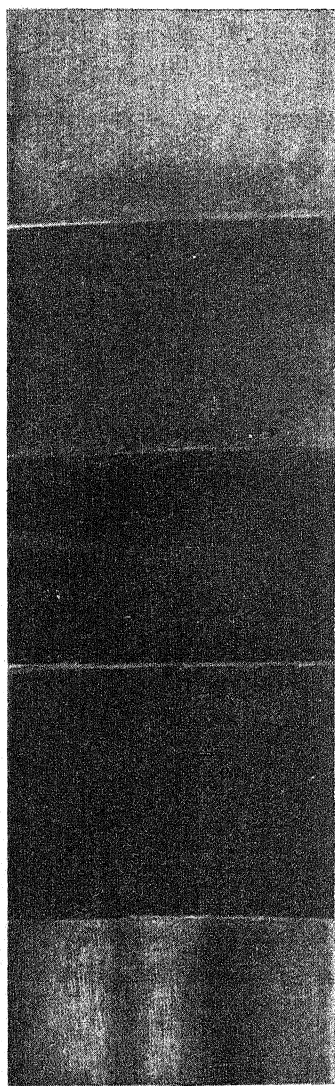
Rayleigh—(Royal Institution lecture on "Polish," March, 1901) also holds the view that the process of polishing is a molecular one.

small granules of oxidised metal formed during heating which differs from the metal itself in its optical properties. As the result of the experiments that will be described in the subsequent pages, it can definitely be asserted that the colours exhibited by heated metal surfaces are due to the *scattering of light* by the granular film of oxide formed during heating.

II.—EXPERIMENTAL DETAILS.

(i) *Preparation of metal plates.*

On putting a polished metal plate upon a Bunsen flame, colours come in such a rapid succession that it is impossible to obtain an uniformly-coloured plate in this way. Moreover, the temperature of the flame being rather high, it is difficult to regulate the heating apart from uniformity, so that there is no knowing as to when to stop in order to obtain a plate, say blue, green or yellow-coloured. With a view to avoid all these difficulties a method of slow heating by an electrical heater was employed. The small metal plate to be experimented upon, was placed on a bigger metallic plate which later was placed as near as possible to the heating coils. The resistance of the coils so employed was 60 ohms and the current was run from the main switch of 220 volts. Thus the heating was slow and uniform and moreover, it gave all facilities for proper regulation. The current was allowed to run through the circuit for 5 minutes before the plates were put in position. In this way different plates were prepared both of copper and iron, which exhibited most of the colours that enter into the composition of white light. As a result of experiment, it is found that uniformity of colour is also ensured when the surface is perfectly polished and clean, quite devoid of any sort of grease. Dirty plates begin to grow high order colours at the places where dirt exists, long before the first traces of colour are seen in other parts of the surface even, when all other precautions for uniform heating are taken. In each case, colour starts at about a reddish or violetish tint which is rather difficult to distinguish on account of the surface colour of the metals. Next, the colour turns to violet in case of copper, and indigo in case of iron. Copper being further heated shows indigo, green, yellow in succession. On still further heating, rather at a high temperature the colour is almost



THE COLOURS OF TEMPERED STEEL.

white and then again it starts from red, and ends in green and yellowish green. Further heating blackens the plate due to the formation of the black oxide. The colours exhibited at this repetition are very rich and gorgeous. In this stage, the formation of the surface-structure becomes visibly discontinuous and granular which, when gently rubbed, falls off here and there exposing the clean metal surface below. The iron plates exhibit beautiful blue and green colour after the indigo state. They next turn to red. The periodic recurrence of colours is beautifully observed in case of copper plates. The conductivity of the metal being high, to obtain all the colour sequences upon the same plate, a temperature gradient was set up along the surface, the far end being allowed to rest upon a block of ice while the other end was put on a Bunsen flame. Thus, bands of colour were made to travel from the hotter to the colder part of the plate on account of this temperature gradient. Such a precaution, however, is not needed in case of iron or steel. If a thin strip of these metals be taken and one end kept in touch with a Bunsen flame, the bands with periodic recurrence of colours begin to travel from the hotter to the colder parts of the plate. In this connection, it must be noted that this recurrence of colours at the second stage with all its enhanced gorgeousness is not to be observed even in case of copper in the process of slow heating. Here, after the yellow stage, the colours lose all their brilliancy. For, although, there is a re-appearance of a dull red and finally before blackening a dull greenish yellow colour, yet they can in no way stand comparison with the colours obtained by heating over a Bunsen flame.

Since the formation of the surface-structure is to be attributed to oxidation of the metal, an estimate of the thickness of the structure may be made if the plates are weighed before and after the operation. With this view the plates were polished on both the sides so as to get rid of any foreign matter that might cling to the rough under-surface. After final polish, the plate was slightly heated to expel water vapour and while hot was put inside a desiccator and allowed to cool there. It was then weighed with a good balance. After the operation also, it was put in the desiccator while hot and allowed to cool as before. From the gain in weight of the plates and also on the assumption that the structure on both sides is of the same thickness and nature, it is easy

to estimate the quantity of metal that has gone into the formation of its oxide. Thence, the dimensions of the plate and density of the metal will give the thickness of the metal layer which is oxidised. Thus it was found that in passing from the initial stage to the last green or yellow stage in the process of electrical heating, the thickness of metal layer affected increases from 5μ to 115μ . Thus the process of the formation of the structure is confined within a very thin layer of the metal surface and the metal below remains as good and pure as ever.

(ii) *Microscopic observation of the Plates.*

That the structure formed upon the surface is not at all continuous, but granular, can be seen at once if the plates be subjected to a microscopic study. The difficulty lies in the fact that the granules having the metal surface at the back cannot be illuminated from below. Light from a high candle-power source incident very obliquely upon the plates, however, serves the purpose of illumination fairly well. The plates prepared during the initial stages of heating reveal a very large number of small particles quite separate from one another but sometimes forming clusters interspersed here and there with particles of bigger size. These bigger particles upon the plates at the initial stages might be due to the presence of traces of greese or to a too close formation of small particles which it is impossible for the microscope to show separately. It is not difficult to measure the size of the granules with the help of a micrometer eyepiece properly calibrated. As the colour of the plates changes during the process of heating, the particles also as revealed by the microscope in case of different plates, go on increasing in dimensions. Their diameter in the initial stages appears to be about 225μ and finally in the greenish yellow stage of slow heating the diameter comes out to be about 600μ . During the second stage of re-appearance of colours the size of the particles becomes rather big, the diameter becoming 9,000 A.U., i.e. about four times the diameter of the particles at the initial stages.

The results of heating some of the copper and iron plates are given in a table (Table I) at the end. It will be found that in case of copper, only up to the first yellow stage the heating was continued and in case of iron, the plates were heated up to the

yellowish green stage only. In column 6, the gain in weight of the plates by heating is entered in while in the last column, the results of microscopic observations are given. The size of the particles are given in terms (ka) where $k=2\pi/\lambda$ (λ being the wave length of light) and a the radius of the particles assuming them to be circular in shape.

III.—DESCRIPTION OF THE PHENOMENA OBSERVED.

(i) *Colour and Polarisation of the Reflected Light.*

If a beam of white light be allowed to fall upon one of the metal plates used in the present investigation, then the colour and polarisation of the reflected light vary with the angle of incidence and the thickness of the film upon the surface, in a very remarkable way. It is instructive to be able to observe the phenomena with the differently coloured films simultaneously, and for this purpose may be used one of the copper or iron plates in which the complete succession of colours has been developed by setting up a temperature-gradient between its ends. One of these plates may be simply held in the hand and the light reflected from it at various angles of incidence may be viewed with or without a nicol held in front of the eye. For quantitative determinations, the plate may be mounted up on a spectrometer table and the reflected light observed through a tube fitted with a nicol which can be taken out or put in position at will.

Viewed with the naked eye the reflected colours are most lively at or about normal incidence, and become less and less saturated as the incidence is made more oblique until finally there is so much white light reflected at the surface of the metal that no colour can be distinguished at all. With the copper plates, colours can be distinguished even at fairly oblique incidence and a remarkable *doubling* of the first coloured band may then be observed. The reason for these effects is easily made out if a nicol be held in front of the eye. Even at small obliquities, the reflected light from the thinner portions of the film shows marked changes in intensity and colour as the nicol is rotated about its axis. The thicker portions of the film on the other hand, show little change at this stage as the nicol is turned round; at more oblique incidences, however, the light reflected from the thicker portions of the film shows striking effects being less intense and more vividly coloured in one posi-

tion of the nicol (principal plane perpendicular to the plane of incidence), and more intense but less strongly coloured in the parallel position. In these two cases, beyond a certain angle of incidence which varies with the thickness of the part of the film under consideration, the *colours observed in the two positions of the nicol are complementary to each other.*

As the nicol is rotated, the shifting to and fro of the coloured bands on the plates may be observed, this effect being most conspicuous on the first coloured band. The relative feebleness of the colours when viewed with the naked eye, at oblique incidences is thus easily understood as due to the complementary colours of both components of polarisation being superposed on each other. The apparent doubling of the first coloured band on the copper plates is also due to this cause.

At very oblique incidences, only the thickest parts of the film exhibit vivid colour when viewed through the nicol.

(ii) *Colour and Polarisation of the Scattered Light.*

It is to be remarked as a very important feature, that the films on the metallic surfaces under consideration *scatter* light very strongly. As a general rule it may be stated, that the colour of the scattered light when observed in directions but little removed from that of the regularly reflected light is complementary to it in colour. The behaviour of the scattered light for various angles of incidence can be very beautifully observed with a metal plate showing the complete sequence of colours as in the experiments described in the previous section. For quantitative observation, the plate may merely be held in the path of a strong beam of light and viewed directly or through a nicol. For more exact work, the spectrometer and a telescope tube fitted with a nicol may be used. It is very important that the plate before heating up should have received a high polish. Scratches and other irregularities upon the surface *reflect* light into the eye and thus tend to obscure the true scattered colours.

The observations of the scattered light may be divided into several headings as follows :—

(a) *Normal Incidence (of unpolarised white light).*—In viewing the scattered light with naked eye, when a beam of unpolarised white light is incident normally upon a metal plate, it is found

that the colour of the light scattered by the film passes through several fluctuations as we move our eye from the direction of the plane of the plate to the direction of the incident beam. As already remarked, the scattered light in directions contiguous to the reflected beam is complementary to it in colour. As we move away however, the colour changes. On examining the scattered light with the nicol, it is found that in directions near about the reflected beam the light is unpolarised and shows no change when the observing nicol is rotated. But in more oblique directions the changes in the intensity and colour of the scattered light becomes noticeable and at a small angle with the plate, the scattered colours show complementary tints on rotating the observing nicol through 90° .

(b) *Light polarised perpendicularly or parallel to the plane of incidence incident normally.*—When a beam of white light polarised either in the plane of incidence or perpendicular to that is incident upon the plate, the scattered light very near to the direction of the reflected beam cannot be quenched in any position of the analysing nicol but it simply shows fluctuations in intensity as the analyser is rotated. But commencing from about an angle of 30° till the surface of the plate is reached, the colours can entirely be quenched by the analysing nicol, which re-appear again on further rotation of the nicol.

(c) *Light polarised in any azimuth (say 45°), incident normally.*—When the incident light is polarised in an azimuth of 45° with the plane of incidence, the phenomena to be observed in the scattered light with the help of the analyser is more or less the same as for light polarised in or perpendicular to the plane of incidence, the quenching of colours with the help of the analyser being more perfect as we come nearer the direction of the surface of the plate.

(d) *Oblique incidences with unpolarised white light.*—As the angle of incidence increases, the beauty and diversity of the phenomena to be observed with the scattered light also go on increasing. The colour-changes become more and more frequent and in certain directions the tints change to the complementary colours when the analysing nicol is rotated through 90° . For very oblique incidences it is found that in certain directions the intensity of light in one position of the observing nicol becomes enormously great in comparison with the light observed in the perpendicular position. In case of thin films, such a direction for maximum polarisation is

found at an angle of about 100° with the direction of the incident beam (incidence about 88°) whereas, for plates containing thicker film with larger particles, such maximum polarisation may be observed in two, three or more directions.

(e) *Light polarised in any azimuth incident obliquely.*—When the incident beam instead of being unpolarised, is polarised in any azimuth, the scattered light, shows the fluctuations of colour and intensity as before. Only the colours to be observed for any suitable position of the nicols become more and more saturated. The scattered light in the field, however, cannot be completely quenched in any position of the nicol although some particular colours may completely be cut off by crossing the nicols, a slight movement of the nicols to the right or to the left causing the reappearance of the same colour. In directions near about the incident and the reflected beams the colour-fluctuation on the rotation of the analyser is very prominent.

In this connection, it is to be mentioned that the complementary nature of the scattered light observed at a very small angle with the surface of the plate, for two positions of the nicol when the light is incident normally upon the plate correspond to the observations of wood¹ and later on of Rayleigh² in case of thin film of collodion spread upon metal plates.

(iii) *Quantitative Study of the Scattered Light.*

All measurements and observations in connection with the experiments to be described in this section were made with a Cornu polarimeter, which is generally used in the experiments for the analysis of the polarised light. Here, the telescope tube consists of a small rectangular opening through which the light to be examined passes, a double-image-prism, and a nicol capable of being rotated about its own axis and attached to a divided circle. The double-image prism is fixed at such a distance from the rectangular opening that the two images—one polarised vertically and other horizontally—are seen one above the other. In the observation of the colour fluctuations of the scattered light as described in the previous section, the analyser is dispensed with and

¹ Wood—Physical Optics, page 172, ed. 1914.

² Rayleigh—Phil. Mag. Vol. 34, Nov. 1917.

Scientific Papers, Vol. 6, p. 508.

the colours of the two images formed by scattered light for different positions of the observer's eye as recorded by the graduated circle of the polarimeter upon which the telescope moves are observed and noted. As remarked before, in order to be able to follow the behaviour of the scattered light for quite a wide range a very oblique incidence was used in all the quantitative studies. Results for the observations on various plates and also for different directions of observation at intervals of 10° are entered into a table (Table II) at the end of the paper. The different phenomena roughly detailed in the last section will be found rather precisely put into this table. The angle θ which has been entered into the first column of this table gives the angular measure for the direction of observation commencing from the negative direction of the incident beam so that we approach to $\theta=180^\circ$, as we approach the direction of the surface of the plate when the incidence is very oblique. The next two columns give the colour for the vertical and the horizontal components of the scattered light as formed with the double image prism.

In making the following observations, the analysing nicol of the telescope of the polarimeter was put in position. The incident light was rendered monochromatic with suitable light filters. Observations were made with three different kinds of filters, viz.

$$\lambda=6290 \text{ A.U.}, \quad \lambda=5830 \text{ A.U.} \text{ and } \lambda=4380 \text{ A.U.}$$

for the purpose of comparison. It was found that the results in all three cases agree on the whole, there being a shift in the positions of neutral points and in the direction of maximum polarisation as far as we should expect for the difference of wave length. So, the final graphs were plotted with the results obtained in using the monochromater for which $\lambda=5830 \text{ A.U.}$

To start with, the zero reading of the nicol is made to correspond to its position when one of the images (say the upper one) vanishes. Then the nicol may be set in two positions on the two sides of this zero position for which the intensity of the two components will be equal. Thus, if w_1 and w_2 be the two readings for the angle of rotation about the zero position for which the intensities I_1 and I_2 of the two components are equal, we have

$$\frac{I_1}{I_0} = \frac{\cos^2 w_1}{\sin^2 w_1} = \frac{\cos^2 w_2}{\sin^2 w_2} = \frac{\cos^2 w_1 + \cos^2 w_2}{\sin^2 w_1 + \sin^2 w_2}$$

whence

$$\frac{I_1 - I_2}{I_1 + I_2} = \cos(w_1 + w_2) \cos(w_1 - w_2) \\ = \cos 2w.$$

where w is the mean of the two angles w_1 and w_2 . For the sake of convenience a quantity P is chosen such that

$$P = 100 \frac{I_1 - I_2}{I_1 + I_2} = 100 \cos 2w.$$

Thus we can determine the value of P in different directions for the scattered light within the limit $\theta = 180^\circ$ to $\theta = 0$.

In the graphs of Figs. 1 and 2 the values of P are plotted for different plates the value of η for which is given near the curve.

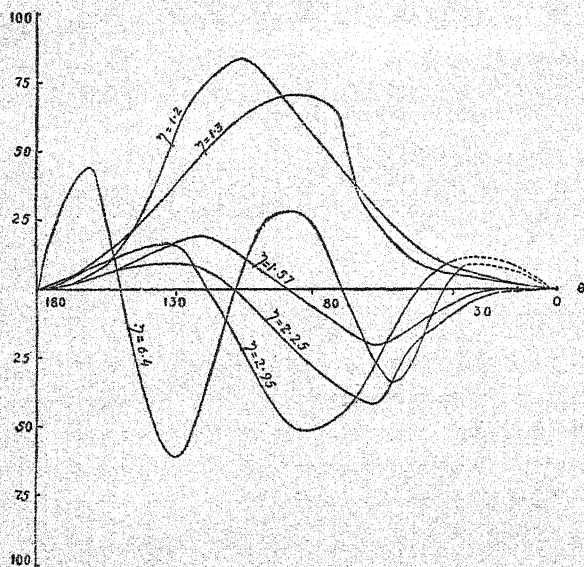


FIG. 1. Experimental curves for Copper.

Taking Fig. 1 which contains the curves for different copper plates it will be seen that for very small particles ($\eta = 1.2$), the maximum polarisation is at an angle of 100° and there is no neutral point ($P = 0$). As the size of the particle increases, the position of the maximum polarisation also is shifted towards the incident light at first. For $\eta = 1.3$, we have the position of maximum polarisation at $\theta = 90^\circ$ and the corresponding value for P is 68, i.e.

smaller than the former value. In the last three curves, viz. $\eta=1.57$, $\eta=2.25$ and $\eta=2.95$, we note many peculiarities. Each of these curves show two or more maxima and neutral points. For $\eta=1.57$, a neutral point is found at $\theta=95^\circ$, which point is found to shift towards the direction of reflected beam as the size of the particles increases. It is also to be noted that the negative values of P are rather prominent than the positive in the latter cases. In the last curve, viz. $\eta=2.95$ we have two neutral points one at $\theta=120^\circ$ and the other at $\theta=45^\circ$. Hence, the experimental curve for iron will explain itself. From these observations on intensity the following are the peculiarities to be observed.

(a) The scattered light is never plane-polarised in any direction, but there is a direction of maximum polarisation for small particles ($\eta=1.2$, $\eta=1.3$), which are formed during the first stage of heating of the plates. For the smallest particles obtained by the author, the position of this maximum is at $\theta=100^\circ$.

(b) As η increases, the position of maximum first moves towards the incident beam (as in $\eta=1.30$) and then it moves back towards the 120° position, for when $\eta=1.6$ it is at about $\theta=120^\circ$ with the incident beam.

(c) At the same time with the increment in the size of the particles, a neutral point ($P=0$) appears which when $\eta=1.57$ is at $\theta=90^\circ$.

(d) Between the neutral point and $\theta=0$, the polarisation is reversed, that is, the horizontal component I_2 is greater than the vertical component I_1 and consequently P is negative.

(e) As the value of η increases, the position of this neutral point moves towards the direction of the reflected beam and newer neutral points originate on the other side.

That the number of neutral points ($P=0$), increases with the size of the particles, will be seen from the curve for $\eta=6.4$ in Fig. 1. This curve was obtained when light was made to be scattered by the beautiful green particles obtained during the second stage of periodic recurrence when one end of a copperplate was heated in a Bunsen flame while the other end was placed upon a block of ice with a view to maintain a temperature gradient. Altogether, we have 4 neutral points in this case, at about $\theta=150^\circ$, $\theta=110^\circ$, $\theta=70^\circ$ and lastly at $\theta=40^\circ$. Beyond that point it was not possible to follow the change in intensity, as there is a general

falling off in intensity. Another peculiarity to be noticed in this case is the fact that maximum value of P both on the negative and on the positive side goes on decreasing as we move from $\theta=180^\circ$ to $\theta=0$.

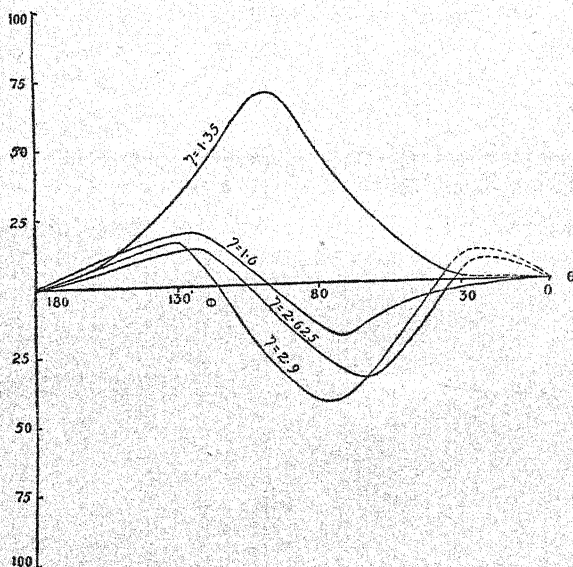


FIG. 2. Experimental curves for Iron.

From a study of the complete set of curves, it is also to be noticed that the maximum value of P obtained with smallest size of particles, is the greatest of all. As the size of the particles increases and they become comparable to the wave length of light ordinary reflexion will take place. As it is a well-known fact that light cannot be perfectly polarised by reflection from a perfectly conducting sphere, finally for all values of θ , the intensity of the two components will be equal or in other words, for very big particles, the curve will reduce to a straight line given by the equation $P=0$.

(iv) *Study of the reflected light with a Babinet compensator.*

Since the condition of the polished surfaces is greatly altered by heating, owing to the formation of the granular film of oxides, it is of interest to determine how the different metallic constants as known for the polished metallic surfaces change, owing to the formation of the structure upon the surface. To a very large

extent, at least in the case of the thicker films these constants would be mainly determined by the optical properties of the film itself and only in a minor degree by the properties of the pure metal surface below; so that the changes in the optical constants may be regarded as indicative mainly of the properties of the surface structure.

The constants were determined by Drude's¹ method from the analysis of the elliptically polarised light formed when plane polarised light is reflected from these surfaces, by means of a Babinet compensator. As in case of the observations recorded in the previous sections, the constants were determined for three different wave lengths, in the red, yellow and indigo part of the spectrum by employing suitable monochromators. As a general rule, it is found that as the nature of the surface-structure changes with heating the apparent index of refraction goes on decreasing whereas the apparent coefficient of absorption k goes on increasing thus giving a consequent increase in reflecting power R of the plates. This was the case with all three wave lengths employed by the author.

The procedure adopted for the experimental determination of the metallic constants is that given in text books on optics. The results were, calculated from the following formulæ of Drude.²

$$(1) \quad k = \tan 2\bar{\psi}$$

where $\bar{\psi}$ = principal azimuth of the analyser

$$(2) \quad n = \frac{\sin \bar{\phi} \cdot \tan \bar{\phi}}{\sqrt{1 + k^2}}$$

where n = refractive Index of the medium.

$\bar{\phi}$ = principal angle of incidence.

$$(3) \quad R = \frac{n^2(1 + k^2) - 2n}{n^2(1 + k^2) + 2n}.$$

The results of calculation are entered in the form of a table (Table III). In column 6 of this table the quantity η has been entered. It is simply the product of the quantity (ka) explained before, with the refractive index of the scattering particles with respect to that of the pure metal surface in which they are embedded. As this quantity η will be required in the theoretical

¹ Drude—Ann. der. Physik. XXXIX (1889).

² Drude—Theory of Optics, pages 363-364. edition 1913.

calculations that follow it has been thought advisable to calculate and record it here. The value of the refractive index for the pure metal required in this transformation of (ka) to η is taken in comparison with the values obtained by Minor.¹

IV.—PHYSICAL THEORY OF SCATTERING BY CONDUCTING PARTICLES.

A solution of the problem of the scattering of electromagnetic waves by a spherical obstacle has been given by Love.² Lord Rayleigh³ has shown that Love's results admit of considerable simplification and has used the simplified forms to calculate the polarisation of the scattered wave when light is incident upon a transparent sphere of refractive index 1.5 and of dimensions comparable with the wave length of light. In the present investigation similar calculations have been made for conducting spheres of copper and iron, the incident beam being a monochromatic wave of length given by $\lambda=5830 \text{ A.U.}$

Let a beam of unpolarised light travelling along the negative direction of axis of Z be incident on a spherical obstacle placed with its centre at the origin. Suppose that the observation on the scattered light are made in the XZ plane at a distance r from the obstacle.

If X, Y, Z be the electric forces parallel to the three axes in the scattered wave then the vertical and horizontal components of intensity in the scattered light are given by the squares of modulus of the complex quantities Y and

$$\frac{xZ - zX}{r}.$$

According to Rayleigh⁴ we get for these two quantities from Love's solution the two following series:—

$$Y = \sum_{n=1}^{n=\infty} (-1)^{n+1} \cdot \frac{2n+1}{n(n+1)} \left[M_n \{ \mu P'_n - n(n+1)P_n \} + N_n P'_n \right] \times \frac{e^{ik(ct-r)}}{kr} \dots (I)$$

¹ Minor—Ann. d. Physik, 4 Folge Bd. X, page 617 (1903).

² Love—Lond. Math. Soc. Proc. Vol. XXX, p. 308 (1899).

³ Rayleigh—Proc. Roy. Soc. Vol. LXXIV, pp. 25-46 (1910).

⁴ Rayleigh—Loc. cit.

$$\frac{xZ - zX}{r} = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{2n+1}{n(n+1)} \left[N_n \{ \mu P'_n - n(n+1)P_n \} + M_n P'_n \right] \times \frac{e^{ik(ct-r)}}{kr} \dots\dots (2)$$

In these equations $k=2\pi/\lambda$ where λ is the wave length of the incident light, $\mu=\cos \theta$ and P_n or $P_n(\mu)$ is a zonal harmonic of degree n whose axis is the axis Z . M_n and N_n are functions of the size and optical properties of the spherical obstacles. And they are given by

$$N_n = \frac{K \cdot \psi_{n-1}(ka) - \left\{ (K-1) \frac{n}{2n+1} + \frac{\psi_{n-1}(\eta)}{\psi_n(\eta)} \right\} \psi_n(ka)}{-KE_{n-1}(ka) + \left\{ (K-1) \frac{n}{2n+1} + \frac{\psi_{n-1}(\eta)}{\psi_n(\eta)} \right\} E_n(ka)} \dots\dots (3)$$

and

$$M_n = \frac{\psi_{n-1}(ka) - \frac{\psi_{n-1}(\eta)}{\psi_n(\eta)} \cdot \psi_n(ka)}{-E_{n-1}(ka) + \frac{\psi_{n-1}(\eta)}{\psi_n(\eta)} E_n(ka)} \dots\dots\dots (4)$$

where K =dielectric constant of the material of the sphere

$$\psi_n(ka) = (-1)^n \cdot 1 \cdot 3 \cdot 5 \dots (2n+1) \left\{ \frac{1}{(ka)} \cdot \frac{d}{d(ka)} \right\}^n \cdot \frac{\sin(ka)}{(ka)}$$

$$E_n(ka) = (-1)^n \cdot 1 \cdot 3 \cdot 5 \dots (2n+1) \left\{ \frac{1}{(ka)} \cdot \frac{d}{d(ka)} \right\}^n \cdot \frac{e^{-ika}}{(ka)}$$

and a =radius of the spherical obstacle.

$\eta=m(ka)$, where m is the refractive index of the material of the sphere with respect to the surrounding medium.

There is however a further relationship between the functions E_n and ψ_n . Because ψ_n is the imaginary part of the complex function E_n . So that separating the real and imaginary parts

$$E_n(\eta) = \Psi_n(\eta) - i \psi_n(\eta) \dots\dots\dots (5)$$

Relations similar to (1) and (2) above have been used by Mie¹ to calculate the intensity and polarisation of the light scattered by small gold particles. And it is to be noted that Mie's solution of the problem of the scattering of light by a sphere is identical with Love's, though obtained independently and expressed rather in a different manner.

Hence, to arrive at the expressions for M_n and N_n for a per-

¹ Mie—Ann. d. Physik, 4 Folge, Bd XXV, p. 427 (1908).

fect conductor we may make use of the expressions given by Sir J. J. Thomson in his "Recent researches in electricity and magnetism," page 446, as has been done by Talbot-Paris.¹ The expressions given by J. J. Thomson as referred to above, give the electric forces in the waves scattered by a sphere having the character of a perfect conductor. But these expressions can be utilised only in the case of ordinary conducting metals provided that the distance which the alternating currents generated by the incident vibrations, penetrate into the sphere, is small compared with the diameter of the sphere. This condition, however, cannot be fulfilled by any known metal when we are dealing with oscillations of the frequency of light and can only be expected to be fulfilled in the case of incident Hertzian waves on the spheres of metals like copper.

By making proper alterations in the notations used by Thomson for the electric forces in the scattered wave we get expressions identical with those of Love, the only difference being in the expressions for M_n and N_n . According to J. J. Thomson, for a perfect conductor

$$M_n = -\frac{1}{(ka)^n} \cdot \frac{S_n(ka)}{f_n(ka)}$$

and

$$N_n = -\frac{1}{(ka)^n} \cdot \frac{\frac{d}{da} \left\{ a \cdot S_n(ka) \right\}}{\frac{d}{da} \left\{ a^{n+1} \cdot f_n(ka) \right\}}.$$

where, as before $k=2\pi/\lambda$ and a =radius of the sphere moreover,

$$S_n(ka) = (ka)^n \left\{ \frac{1}{ka} \cdot \frac{d}{d(ka)} \right\}^n \cdot \frac{\sin(ka)}{(ka)}$$

and

$$f_n(ka) = \left(\frac{1}{ka} \cdot \frac{d}{d(ka)} \right)^n \cdot \frac{e^{-ika}}{(ka)}.$$

Hence for the expressions M_n and N_n in terms of E_n and ψ as before, we have,

$$M_n = \frac{\psi_n(\eta)}{-E_n(\eta)} \dots\dots\dots (6)$$

¹ Talbot-Paris—Phil. Mag. Vol. 30, Oct. 1915.

and

$$N_n = \frac{(2n+1)\psi_{n-1}(\eta) - n\psi_n(\eta)}{-(2n+1)E_{n-1}(\eta) + nE_n(\eta)} \dots\dots\dots (7)$$

Hence M_n and N_n can easily be calculated and substituted in the equations (1) and (2) before. Since Y and $\frac{xZ-zX}{r}$ are both complex quantities we may put

$$Y = Y_1 + i Y_2$$

$$\frac{xZ-zX}{r} = Z_1 + i Z_2$$

So that
$$\frac{\text{vertical component of intensity}}{\text{horizontal component of intensity}} = \frac{Y_1^2 + Y_2^2}{Z_1^2 + Z_2^2}$$

This relation can easily be determined experimentally and truth of the assumptions will thereby be established.

V.—THEORETICAL RESULTS FOR COPPER AND IRON PLATES.

In order to calculate the polarisation curves for light scattered by small particles, it is necessary to make use of the general expressions for M_n and N_n already quoted (page 31). In taking in value for η however, the refractive index of metallic copper and of the surface-structure formed by heating are both to be considered. Hence the values of η calculated in the Table III are taken.

The first step towards obtaining the values of M_n and N_n is to calculate the value for $\psi_n(\eta)$ from the series.

$$\psi_n(\eta) = 1 - \frac{\eta^2}{2 \cdot 2n+3} + \frac{\eta^4}{2 \cdot 4 \cdot 2n+3 \cdot 2n+5} \dots\dots$$

Hence, to obtain the values for $\psi_n(\eta)$ for moderate values of n the sequence equation

$$-\frac{\eta^2}{2n+3} \cdot \psi_{n+1}(\eta) = (2n+1) \left\{ \psi_{n-1}(\eta) - \psi_n(\eta) \right\}$$

is employed.

The values of $E_n(\eta)$ are calculated from relations¹

$$E_0(\eta) = \frac{e^{-i\eta}}{\eta}$$

and

$$E_1(\eta) = 3 \cdot \frac{1+i\eta}{\eta^3} \cdot e^{-i\eta}.$$

¹ Rayleigh—*Loc. cit.*

and the sequence equation

$$E_{n+1}(\eta) = \frac{(2n+1)(2n+3)}{\eta^2} \{ E_n(\eta) - E_{n-1}(\eta) \}.$$

Hence since $E_n(\eta) = \Psi_n(\eta) - i\psi_n(\eta)$ the value of both the functions Ψ_n and ψ_n are known. The following table (Table IV) will give the values of these two functions. The values are checked as far as practicable. Next, with the values for Ψ_n and ψ_n and with the help of the relations deduced before, the Table V giving the values for M_n and N_n is made.

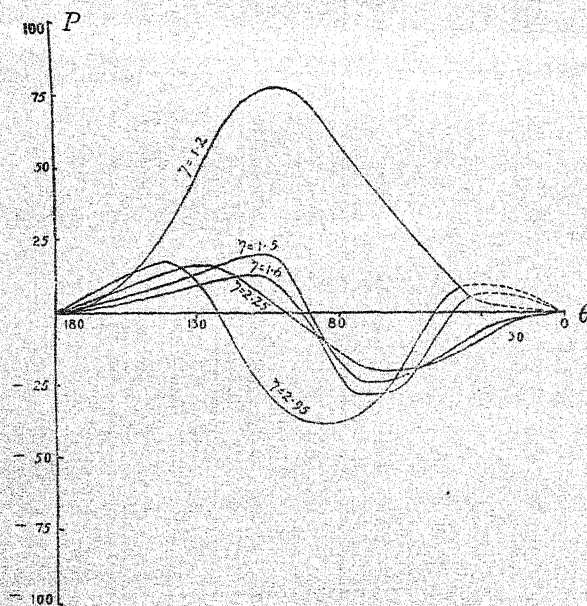


FIG. 3. Theoretical curves for Copper and Iron.

The logarithmic values of

$$\frac{(2n+1)P'_n}{n(n+1)} \quad \text{and} \quad \left\{ \frac{(2n+1) \cdot \mu P'_n}{n(n+1)} - (2n+1)P_n \right\}$$

have been tabulated by Rayleigh¹ for the five values of μ , viz. $\mu=0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ and 1. Thus the expressions for the vertical and horizontal components of the electrical force in the scattered waves may be calculated for different directions given by the values of $\mu = \cos \theta$. When μ is negative $\frac{(2n+1)P'_n}{n(n+1)}$ changes sign if n is even and the other quantity changes sign if n is odd. Hence the values

¹ Rayleigh—*Loc. cit.*

for the squares of the moduli are calculated for five different values of η and the value of P obtained in these cases also. The results are entered into a table (Table VI).

The values of P are plotted against the corresponding values of θ (Fig. 3) making use of the same units as in the case of Figs. 1 and 2.

In comparing the two sets of curves, one obtained experimentally, (Figs. 1 and 2), and the other calculated from the theory (Fig. 3), it will be found that for the smallest size of particle, viz $\eta=1.2$, the two curves are almost similar, there being a direction of maximum polarisation between $\theta=100^\circ$ to 90° . In the curve for $\eta=1.6$ the position of the neutral point is almost the same. But the value for maximum value of P both on the negative and on the positive sides of the axis is greater in the case of the theoretical curve than in the experimental curves. The error may be due to the estimation of the intensity—equalisation in those positions. There are such discrepancies in case of the other curves which lie within the limit of experimental error and on the whole the two sets of curves may be considered identical. The calculations in case of such large particles as given by $\eta=6.4$ was not tried on account of the tediousness of the task and the larger calculation that it will involve. For, the calculations in case of the smaller particles were made easy by the calculated results of Rayleigh for the factors which finally multiply M_n and N_n . Thus, the theoretical calculations were restricted only to five cases. Thus the phenomena observed and recorded in the preceding pages may be accounted for by considering the cause as the scattering of light by the close-grained discontinuous structure formed upon the polished metallic surface in the process of heating beyond the tempering range. As the scattered light is distributed over quite a wide range its colour changing continually, we should expect to find the scattered light in the directions in which it is brightest to be complementary to the reflected light. To account for the non-changeability of the colour as the tempered surface is gently polished, the argument may be put forward that since the small particles playing the whole part are of dimensions small compared with the wavelength of light, they will be removed from the surface wholly and never in parts, however gentle the touch of the polishing apparatus may be. Thus by polishing

gently, we simply reduce the number of particles distributed over the area, which is the cause for falling off in intensity of the light observed by Mallock. Moreover, this conception of granular structure and the scattering by the small particle fits in closely with the phenomena of periodic recurrence of colours observed both in case of copper and iron.

That the time of heating plays a part in these cases, there is no doubt. But the fact that the second stage of re-appearance of colours could not be obtained unless the end of the copper plate was put in contact with the Bunsen flame itself cannot be ignored. As a sort of explanation it may be put forward that as heat is applied to the plate, the molecules lying at random (the idea of Beilby) are not affected all at once. If the temperature is kept constant, more and more molecules will be caused to take part in the process and some will form aggregates forming bigger particles which account for the gradual change in colour. That the colours may be obtained even at quite a low temperature (100°C) may be seen on looking at the inner surface of the double-walled copper-made water-baths used so often for keeping a thing at a constant temperature. If the surfaces are sufficiently clean and polished, almost all the colours will be found there in quite a beautiful manner. The author of the present paper however, put a polished copperplate inside one of such water-baths. After an operation for about a month, the water being heated for about five hours daily, a violet tint was obtained upon the plate. That longer exposures will bring in other colours is quite apparent from the colours upon the body of the water-bath itself. But here the metal is being heated in contact with the water vapour and hence the re-action taking place in the formation of aggregates is of quite a different nature from that under investigation.

Again, by this slow heating the degree and nature of oxidation of the molecules will also go on changing so much so that on continuation of heating at the same temperature which is rather low, the particles instead of forming into still larger aggregates necessary for the colours in the second stage, will gradually pass on to form the black-oxide. Hence it was that the high temperature of a Bunsen flame was necessary to form larger aggregates long before the nature of the particles was changed to the complete oxidation stage.

VII.—SUMMARY AND CONCLUSION.

(1) In the present paper an attempt has been made to explain the colours exhibited by tempered steel and other tarnished metal surfaces, the diffraction of light by the granular film of oxide formed during tempering playing the main part in the theory on the lines first suggested by C. V. Raman.

(2) The experiments made with differently heated plates of different metals supply strong evidence that the colours are due to the granules.

(3) The experiments done consist in—

(i) Observation and study of the reflected light as the angle of observations are changed and also of the scattered light in general.

(ii) Study, in detail of the scattered light, for various angles between $\theta=0$ to $\theta=180^\circ$ with quite a large angle of incidence.

(4) Curves are drawn showing the intensity relations of the vertical and the horizontal components in the scattered light for values of θ ranging between 0° to 180° .

(5) The physical theory of scattering as developed by Love and Rayleigh has been employed with slight modifications for conducting particles as has been done by Talbot-Paris.¹

(6) The theoretical intensity curves are drawn employing different values for the size of the scattering particles which is determined by the microscopic study of the structure formed upon the metal plates.

It is found that the curves obtained experimentally and from theory agree quite remarkably.

Thus the idea that the scattering of light by the surface structure is an essential part of the phenomena is supported by the detailed observations that have been made and recorded in course of this paper. The theory also explains in a general way the facts in connection with the problem known before and observed by others.

The author wishes to record his cordial thanks to Prof. C. V. Raman for his great interest in the work and for providing all facilities for work at the Palit Research Laboratory.

¹ Talbot Paris—*loc. cit.*

VIII.—EXPLANATION OF THE PHOTOGRAPH.

The photograph shows the colours in different plates of iron that have been employed in the present investigation. The last figure shows the recurrence of colours on the same plate while the other plates exhibit the colour for heating for different periods as recorded in Table I.

*University College of Science,
Calcutta.
3rd May, 1922.*

TABLE I.

Metal plate.	Dimensions of the plate in sq. cm.	Time for which the current was switched on.	Colour exhibited.	Original weight of the plates in gm.	Gain in weight after heating (in gms.).	Size of the particles (μ).
<i>Copper.</i>						
I ..	6.2 x 3.5	1½ min.	Red tint	16.303	0.0002	1.21
II ..	6.1 x 3.8	2 min. 20 sec.	Violet	15.470	0.0008	1.33
III ..	6.8 x 4.0	2 min. 40 sec.	Green tint	15.262	0.001	1.64
IV ..	6 x 3.7	3 min.	Green	15.9458	0.0031	2.4
V ..	6.8 x 4	3½ min.	Yellow	15.562	0.005	3.196
<i>Iron.</i>						
I ..	7 x 4	3 min.	Indigo tint	24.9412	0.0008	1.423
II ..	6.8 x 4	3½ min.	Indigo	25.591	0.001	1.691
III ..	6.9 x 3.8	5 min.	Blue	23.568	0.004	3.054
IV ..	6.9 x 4.6	10 min.	Green (yellowish)	28.271	0.007	4.115

TABLE II.

θ	COPPER PLATE I.	
	Vertical Component.	Horizontal Component.
160°	Greenish yellow	Bluish green
150	Yellow	Green
140	"	"
130	"	"
120	Reddish yellow	Yellowish green
110	"	"
100	Yellow	Green
90	"	"
80	Greenish yellow	"
70	"	"
60	"	"

θ	COPPER PLATE II.	
	Vertical Component.	Horizontal Component.
160	Greenish yellow	Greenish yellow
150	"	"
140	Reddish yellow	"
130	"	"
120	"	"
110	Yellowish red	Yellowish green
100	"	"
90	"	"
80	Reddish yellow	Green
70	"	"
60	"	"

θ	COPPER PLATE III.	
	Vertical Component.	Horizontal Component.
160	Greenish yellow	Red
150	"	"
140	"	Reddish yellow
130	"	"
120	Yellowish green	"
110	"	"
100	Yellow	Yellow
90	Greenish yellow	Reddish yellow
80	"	"
70	"	"
60	"	Yellowish red

θ	COPPER PLATE IV.	
	Vertical Component.	Horizontal Component.
160	Yellow	Red
150	"	"
140	Greenish yellow	Yellowish red
130	"	"
120	Yellowish green	Reddish yellow
110	"	"
100	"	"
90	Green	Yellow
80	Yellowish green	Reddish yellow
70	Yellow	"
60	"	Red

θ	COPPER PLATE V.	
	Vertical Component.	Horizontal Component.
160	Reddish yellow	Bluish green
150	"	"
140	Greenish yellow	Blue
130	"	"
120	"	"
110	Yellow	Yellowish green
100	"	Blue
90	Reddish yellow	Greenish yellow
80	"	Green
70	"	"
60	"	"

TABLE II—*contd.*

°	IRON PLATE I.	
	Vertical Component.	Horizontal Component.
160	Yellow	Greenish yellow
150	"	Green
140	"	"
130	"	"
120	Greenish yellow	Yellowish green
110	"	"
100	"	"
90	"	"
80	"	"
70	"	"
60	"	"

°	IRON PLATE III.	
	Vertical Component.	Horizontal Component.
160	Yellow	Yellow
150	Greenish yellow	Reddish yellow
140	"	"
130	"	"
120	"	"
110	"	"
100	Yellow	"
90	"	Yellow
80	Greenish yellow	Reddish yellow
70	"	"
60	"	"

°	IRON PLATE II.	
	Vertical Component.	Horizontal Component.
160	Yellow	Yellow
150	"	"
140	"	Green
130	"	"
120	Greenish yellow	Yellowish green
110	"	Green
100	"	"
90	"	"
80	"	"
70	"	"
60	"	"

°	IRON PLATE IV.	
	Vertical Component.	Horizontal Component.
160	Reddish yellow	Greenish yellow
150	"	"
140	Yellow	"
130	Greenish yellow	Reddish yellow
120	"	"
110	"	"
100	Yellow	Yellow
90	"	"
80	Greenish yellow	Reddish yellow
70	"	"
60	"	"

TABLE III.

Metal plate.	$\lambda \times 10^3$ cm.	n (Refractive index).	k =Co- efficient of absorp- tion.	R (Reflect- ing power).	$\eta = m(ka)$ where $m = \frac{n}{n'}$.	n' = refrac- tive index of the polished surface.
Copper- plate I.	6290 5830 4380	0.571 0.644 1.098	4.705 4.0111 2.050	75%	1.2	0.650
Copper- plate II.	6290 5830 4380	0.562 0.637 1.08	5.670 4.718 2.246	78.6%	1.3	
Copper- plate III.	6290 5830 4380	0.552 0.620 1.075	7.115 5.673 2.475	86.9%	1.57	
Copper- plate IV.	6290 5830 4380	0.500 0.608 1.07	9.514 7.125 2.605	91.5%	2.25	
Copper- plate V.	6290 5830 4380	0.496 0.600 1.060	14.300 9.534 2.747	96%	2.95	
Iron plate I.	6290 5830 4380	1.727 2.352 2.527	1.664 1.483 1.428	50%	1.35	2.480
Iron plate II.	6290 5830 4380	1.663 2.346 2.422	1.732 1.600 1.428	53.7%	1.6	
Iron plate III.	6290 5830 4380	1.552 2.132 2.363	1.732 1.732 1.588	54%	2.625	
Iron plate IV.	6290 5830 4380	1.480 1.748 2.132	1.881 2.104 1.732	54.5%	2.90	

TABLE IV.

$\eta=1.2$ (COPPER PLATE).			$\eta=1.5$ (COPPER PLATE).		
n	Ψ_n	ψ_n	n	Ψ_n	ψ_n
0	.30205	0.77650	0	.04716	0.66499
1	2.57051	0.86298	1	1.3929	0.79235
2	23.630	0.90092	2	8.9713	0.84900
3	510.85	0.92232	3	117.89	0.88121
4	21315.	0.93608	4	3049.3	0.90204
5	—	0.94592	5	—	0.91664
6	—	0.97695	6	—	0.92743

$\eta=2.95.$		
n	Ψ_n	ψ_n
0	—33550	0.06462
1	—04892	0.36070
2	+61520	0.51039
3	+2.6707	0.60218
4	14.876	0.66456
5	138.81	0.70985
6	2035.5	0.74435

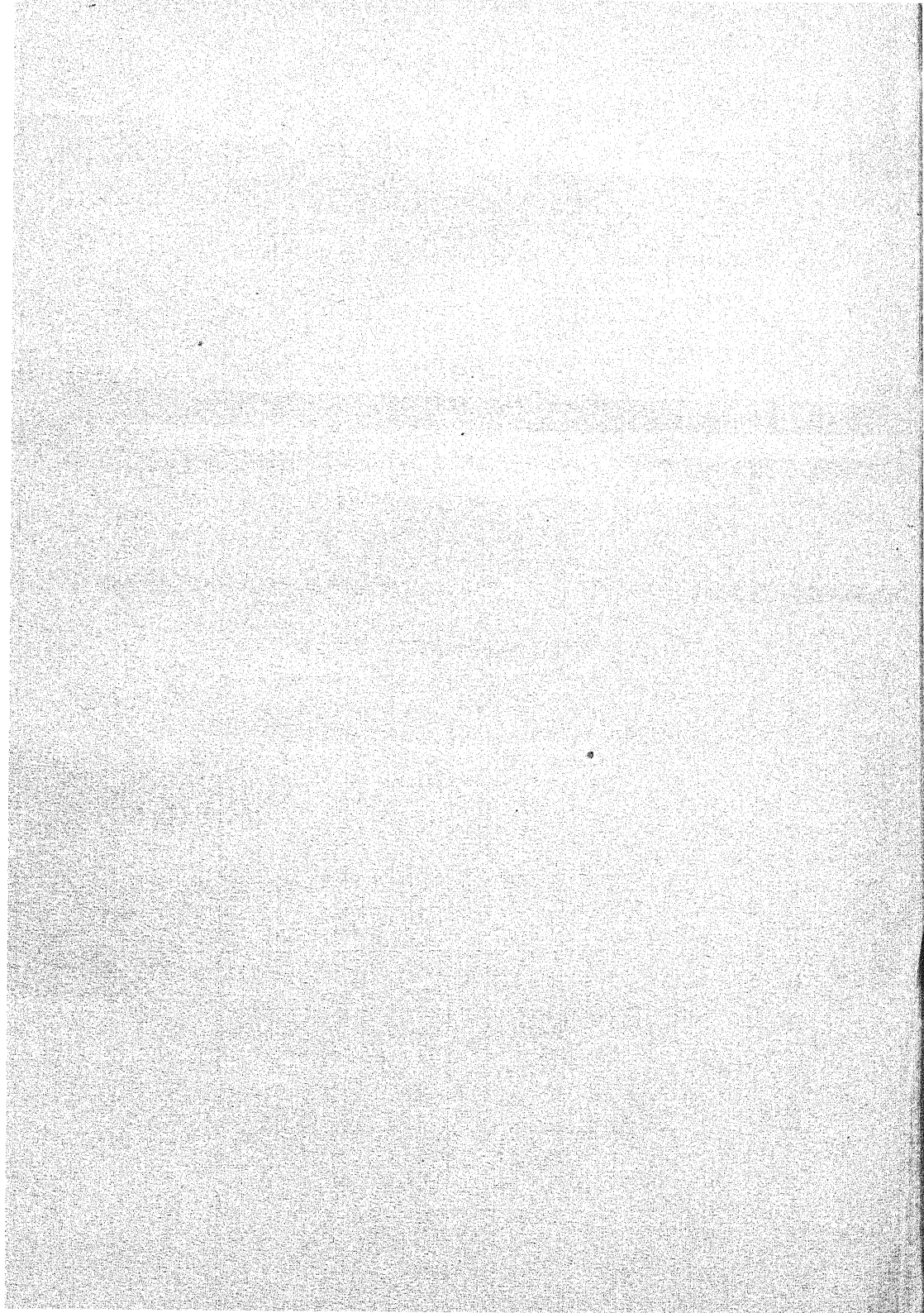
$\eta=1.6.$			$\eta=2.25.$		
n	Ψ_n	ψ_n	n	Ψ_n	ψ_n
0	—01798	0.61702	0	—27918	0.34581
1	+1.17906	0.75697	1	+29563	0.57717
2	70.141	0.82029	2	1.70318	0.68552
3	942.76	0.86578	3	9.73119	0.74909
4	21459.	0.88920	4	99.901	0.79107
5	—	0.90563	5	1933.5	0.82092
6	—	0.91781	6	51793.7	0.84327

TABLE V.

n	M_n	N_n
$\eta = 1.2.$		
1	$-0.30172 - i \times .10128$	$0.49602 - i \times .29812$
2	$-0.03807 - i \times .00145$	$0.09536 - i \times .010126$
3	$-0.00181 -$	$0.00259 -$
$\eta = 1.5.$		
1	$-.42994 - i \times .24458.$	$0.49944 - i \times .39935$
2	$-.09877 - i \times .009768$	$0.22756 - i \times .05466$
3	$-.00746 - i \times \text{---}$	$0.01134 - i \times .000128$
4	$-.000296 -$	0.00038
$\eta = 1.6.$		
1	$-.45484 - i \times .29201$	$0.49545 - i \times .56793$
2	$-.01169 - i \times .000136$	$0.01596 - i \times .00025$
3	$-.000916$	0.00116
$\eta = 2.25.$		
1	$-.40579 - i \times .07921$	$.40044 - i \times .14169$
2	$-.34645 - i \times .13948$	$.48574 - i \times .38161$
3	$-.07652 - i \times .00589$	$.16066 - i \times .02066$
4	$-.00791 - i \times \text{---}$	$.00676 - i \times .000045$
5	$-.000424$	$.00053$
$\eta = 2.95.$		
1	$+ .13320 - i \times .98212$	$- .016908 - i \times .002944$
2	$- .49182 - i \times .40798$	$+ .41619 - i \times .22085$
3	$- .21455 - i \times .04831$	$+ .38833 - i \times .18510$
4	$- .04446 - i \times .001986$	$.07787 - i \times .00606$
5	$- .00511 - i \times .000026$	$.00709 - i \times .00005$
6	$- .000363$	$.000437$

TABLE VI.

$\mu = \cos \theta$	$\eta = 1.2.$			$\eta = 1.5.$		
	$Y_1^2 + Y_2^2$ (I_1).	$Z_1^2 + Z_2^2$ (I_2).	$P.$	$Y_1^2 + Y_2^2.$	$Z_1^2 + Z_2^2.$	$P.$
1	0.8461	0.8461	0	0.9287	0.9287	0
0.9	1.716	1.2705	14.92	0.6595	0.8146	-10.5
0.8	1.194	0.5963	33.01	0.9040	1.313	-18.45
0.7	0.8486	0.2142	59.84	1.1276	1.4403	-12.16
0.6	0.6206	0.0777	77.73	1.3011	1.5110	-3.70
0.5	0.3328	0.0598	69.53	1.3079	1.237	+2.78
0.4	0.4405	0.1082	60.54	1.407	0.9381	+20.00
0.3	0.3895	0.2067	29.71	1.4378	1.0604	+15.00
0.2	0.5855	0.5855	0	2.461	2.461	0
				$\eta = 2.95.$		
				$Z_1^2 + Z_2^2.$	$P.$	
1				2.0289	0	
0.9				1.8447	+6.5	
0.8				3.4832	-26.57	
0.7				3.7880	-35.57	
0.6				5.6497	-36.84	
0.5				5.2506	-35.61	
0.4				3.0610	+7.29	
0.3				6.2855	+18.13	
0.2				15.79	0	
				$\eta = 2.25.$		
				$Y_1^2 + Y_2^2.$	$Z_1^2 + Z_2^2.$	$P.$
1	1.0237	1.0237	0	1.2095	1.2095	0
0.9	0.687	0.808	-7.09	4.3414	3.9872	+4.253
0.8	1.1389	1.381	-9.5	1.0658	1.7811	-18.6
0.7	0.8618	1.4137	-22.80	.6429	0.7625	-8.4
0.6	0.9092	1.0986	-9.4	1.3333	1.4000	-2.4
0.5	1.2927	0.8122	+21.78	0.8268	0.6010	+15.8
0.4	1.288	1.0239	+15.5	1.8043	1.2048	+19.9
0.3	1.981	1.562	+12.2	2.1407	1.7860	+8.5
0.2	2.659	2.659	0	6.3199	6.3199	0



XI. Thunderstorms in Trivandrum.

By K. R. Ramanathan, M.A., Director of the Trivandrum Observatory.

I.—INTRODUCTION AND SUMMARY.

Thunderstorms are a regular feature of Trivandrum weather. They present well-marked seasonal and diurnal variations. The maximum activity occurs during the months March to May and in October. In the following paper, an analysis is made of the seasonal variations of thunderstorms in Trivandrum and they are discussed together with the variations of other meteorological elements, humidity, temperature and air movement both at Trivandrum and at Augustia (6,200 ft. above sea level) in the light of Simpson's theory of thunderstorm formation. According to Simpson, whenever there are strong humid, ascending air currents in the atmosphere accompanied by condensation of moisture, splitting of water-drops and separation of electricity occur. The conditions that have been recognized as essential for the formation of strong humid ascending air currents are (1) sufficient moisture in the atmosphere and (2) a vertical temperature-gradient exceeding the adiabatic lapse-rate. So far as these two factors go, they are satisfied at Trivandrum at all the months of the year. The reason why, then, thunderstorms do not occur at all seasons is shown to be connected with the existence of strong horizontal winds at and above the level of the Western Ghats in the months June to September and November to February, and the comparative absence of such steady air movements during the rest of the year. Strong horizontal winds prevent the formation of strong ascending currents.

II.—STATISTICS OF THUNDERSTORMS.

Observations of thunderstorms in Trivandrum made during the years 1856-1864 by Mr. J. A. Broun have been discussed in

the Indian Met. Memoirs Vol. X, part 1 by Sir J. Eliot. Further records have been kept in the Trivandrum observatory from 1892 onward, when they were begun by Dr. A. C. Mitchell.

Table I gives the number of days on which thunder was heard in the different months of the years 1902-1914.

Table II gives the number of days on which lightning was seen with or without thunder.

TABLE I. *Number of days on which thunder was heard.*

Year.	Jan.	Feb.	Mar.	Apr.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.	TOTAL.
1902	3	1	13	28	23	10	0	0	6	16	14	8	122
1903	2	2	6	26	17	10	0	0	2	11	8	7	91
1904	2	2	9	19	13	8	4	0	0	22	9	1	89
1905	0	4	4	23	15	2	0	0	2	10	9	0	69
1906	0	5	7	14	19	3	0	3	1	20	13	8	90
1907	2	1	17	22	21	5	2	1	14	19	12	7	113
1908	7	8	14	17	12	6	1	0	1	8	7	2	83
1909	1	7	14	22	11	1	1	2	0	14	14	7	94
1910	1	6	13	18	12	5	4	3	1	11	15	0	89
1911	0	1	12	6	17	5	1	0	3	15	8	10	78
1912	1	4	7	24	22	6	0	0	3	14	10	0	101
1913	3	4	5	20	21	4	1	2	2	10	10	6	88
1914	2	3	13	19	16	8	1	2	5	14	11	7	101
Average	2	3	10	20	17	6	1	1	3	15	11	5	93

TABLE II. *Number of days on which lightning was seen with or without thunder.*

Year.	Jan.	Feb.	Mar.	Apr.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.	TOTAL.
1902	3	2	16	29	30	14	0	2	8	22	18	11	155
1903	2	4	10	30	22	12	0	0	2	15	15	9	121
1904	2	6	14	22	22	9	4	1	2	22	9	4	117
1905	0	10	13	26	26	2	0	0	5	11	16	4	113
1906	7	3	12	15	20	5	0	7	4	22	13	10	118
1907	2	7	26	25	25	8	5	1	5	24	15	7	150
1908	9	11	17	20	15	6	2	0	4	10	7	3	104
1909	1	10	16	23	18	3	1	2	0	18	19	8	119
1910	1	6	14	19	14	11	5	3	1	14	17	0	105
1911	2	2	16	10	19	8	1	1	6	15	16	12	108
1912	2	7	10	28	28	19	1	3	10	15	16	3	142
1913	4	6	6	24	25	8	1	2	8	14	17	7	122
1914	3	4	18	22	18	9	1	2	10	21	16	9	133
Average	3	6	14	23	22	9	2	2	5	17	15	7	124

It will be observed that the maximum thunderstorm activity occurs during March to May. There is also a secondary maximum in October. The distribution of thundery weather during the year is shown graphically in Figure 1.

An examination of the records also shows that the thunderstorms begin either to the north or east of Trivandrum, except just previous to the burst of the monsoon or when depressions are

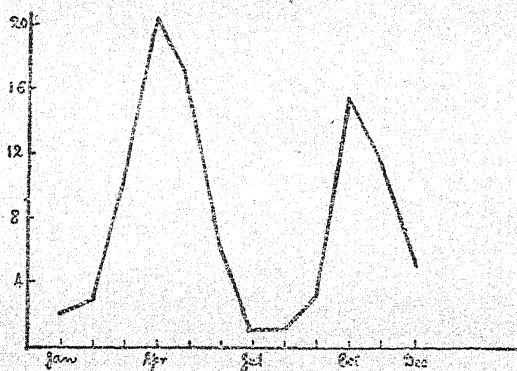


FIG. 1.

travelling close to Trivandrum (when they may begin in any direction) and that they are most active during the afternoon hours 2 to 5 p.m.

III.—MODE OF FORMATION.

The most satisfactory theory of thunderstorm formation is that due to Dr. G. C. Simpson.¹ He showed that splashing of water drops by air currents produces electrification, positive on the drops and negative in the air. When there is opportunity for a large breaking up of water drops, a large quantity of electrification is produced. In the region of humid ascending atmospheric currents, there would be condensation of moisture due to adiabatic cooling, and if the ascending currents are strong, the condensed drops would not be allowed to fall down but would be carried up until they reach a certain size and thereafter their shape would get so much out of the spherical that they would become unstable and break up into smaller drops. Dr. Lenard has shown that the limiting size of waterdrops beyond which they cannot grow without breaking up is about 5 millimetres in diameter and the limiting velocity which drops of this size would acquire on free fall is nearly equal to 8 metres per second. Hence, no drops could fall through an ascending current of air with a vertical velocity greater

than 8 metres per second. The drops would be carried up, reach a certain size, break up, and again be carried up, grow, and break up. Dr. Simpson showed that enough electricity could be produced on the drops by such splashing as would be sufficient to account for the electricity carried down by rain.

Accepting this theory, we shall see how the conditions are favourable for the formation of thunderstorms in the months of March, April and May and in October and November, and how they are unfavourable during the other months.

The conditions favouring strong ascending currents accompanied by large condensation are:—

1. Enough moisture in the lower air for clouds to form as a result of upward movement;

2. A rate of fall of temperature with height approaching or exceeding the adiabatic lapse rate for saturated air; and

3. Absence of strong horizontal winds for a few kilometres above the earth's surface.

I. We shall take these points one by one.

Table III gives the actual pressure of the vapour present in the atmosphere in the different months of the year at Trivandrum and at Augustia peak (Lat. $8^{\circ} 37' N.$, Long. $77^{\circ} 30' E.$, height 6,200 ft. above sea level and distant 22 miles from Trivandrum, a high solitary peak in the Western Ghats where an Observatory was maintained by the Government of Travancore during the years 1856–1858 and in 1864 under the direction of Mr. J. A. Broun for taking magnetic and meteorological observations).

They are taken from the Indian Met. Memoirs Vol. X, parts I and II and are based on 24 hourly observations of the wet and dry bulb thermometer during the years 1856–1864 at Trivandrum and 1856–1858 at Augustia.

Month.	TRIVANDRUM.		AUGUSTIA (height 6,200 ft.).	
	Mean vapour pressure in inches of mercury.	Humidity. %	Mean vapour pressure in inches of mercury.	Humidity. %
January	0.69	74	0.39	93
February	0.68	71	0.38	83
March	0.78	75	0.45	89
April	0.83	77	0.50	90
May	0.87	82	0.49	97

Month.	TRIVANDRUM.		AUGUSTIA (height 6,200 ft.).	
	Mean vapour pressure in inches of mercury.	Humidity. %	Mean vapour pressure in inches of mercury.	Humidity. %
June ..	0·82	86	0·49	98
July ..	0·79	86	0·48	99
August ..	5·79	86	0·47	97
September ..	0·77	83	0·46	97
October ..	0·79	85	0·46	97
November ..	0·77	83	0·44	97
December ..	0·68	76	0·39	94

So far as moisture is concerned, there is plenty of it in all seasons of the year. Indeed, it is doubtful whether there are many other places on earth where there is so much of moisture at all seasons of the year.

II. It is not possible to get an accurate idea of the lapse rate of temperature at Trivandrum without observations with sounding balloons. We can, however, get much useful information from a consideration of the mean temperatures at Trivandrum and Augustia during the different months of the year.

Table IV, gives the mean temperatures at Trivandrum and Augustia based on 24 hourly observations; 1856-64 at Trivandrum, and 1856-58 at Augustia.

Month.	Mean temperature at Trivandrum in °C.	Mean temperature at Augustia in °C.	Difference in °C.
January ..	24·6	12·0	12·6
February ..	25·6	13·5	12·1
March ..	26·9	15·3	11·5
April ..	27·0	16·4	10·6
May ..	26·7	15·2	11·5
June ..	25·3	14·8	10·5
July ..	24·8	14·3	10·5
August ..	24·8	14·2	10·6
September ..	25·0	14·0	11·0
October ..	25·0	14·0	11·0
November ..	25·0	13·6	11·4
December ..	24·7	12·2	12·5

The minimum difference occurs during the months of June and July and even in those months, the lapse rate is 0·36° C per 100 metres (6,200 ft.=1,890 metres) a quantity far exceeding the adiabatic lapse rate for saturated air. According to Hann, the

temperature gradient for 100 metres for saturated air at 25° C under conditions of dynamical equilibrium to a height of 2,000 metres from sea level is 0.43° C per 100 metres (Handbuch der Meteorologie, page 182). We have no data as regards temperatures higher up. So far, then, as the first two conditions for ascensional movement are concerned, they are satisfied in all parts of the year. The answer to the question why thunderstorms do not occur in all months of the year, will be clear when we have discussed the third condition.

IV.—CHARACTER OF THE AIR MOVEMENT IN TRIVANDRUM.

The character of the air movement in Trivandrum is affected to a great extent by the close neighbourhood of the Western Ghats. For about 20 miles to the east of Trivandrum, the country is undulating with hills and hollows, and beyond it, rises the Western Ghats to an average height of 5,000 ft. During the months June to September, when the S.W. monsoon is in full swing, there is a strong steady wind from about N. 60° W. with little diurnal variation. During the months November to April, the air movement consists of land and sea-breezes. The N.E. winds, that obtain during this period in the south of the Peninsula to the east of the Ghats, do not penetrate into Trivandrum, sheltered as it is, by the protecting effect of the Ghats. The months May and October are months of transition.

Since the air movement at Trivandrum is largely affected by the proximity of the high Western Ghats, it is more useful to consider the air movement at a higher level where the winds would be less hampered by geographical peculiarities. For this purpose, we shall consider the air movement at Augustia. In his discussion of Augustia meteorological observations, Sir J. Eliot summarises the general character of the air movement thus:—

“The air movement at Augustia differs essentially in many respects from that prevailing at Trivandrum. The peak is the highest point of the South Travancore Hills.

In the months of December, January and February, when steady and moderate strong N.E. winds obtain in the south-west of the Bay and are continued as E.N.E. winds across the districts of Tinnevely and Madura and when light local land and sea breezes obtain in the Travancore Coast districts and the neigh-

bouring sea area, the air movement at Augustia is determined by (and is a continuation of) the massive atmospheric current from the N.E. over the Bay which is strongest in the S.E. of the Bay and Southern India. In this season, 5% of the wind observations are of Calms, 64% of the winds are from East and 21% from N.E. and only 10% from other directions. The air movement is hence remarkably steady and is on the mean of all the data, from E.N.E., or more exactly, N. 80° E.

“Similar conditions obtain during the third period (June to Sept). The air movement at Augustia is then determined by the S.W. monsoon air current over the Arabian Sea. During this period, in the years 1856-58, about 4% of the observations were of Calms, 21% were winds from N.W. and 72% from W. and hence only 3% from the remaining six points. Winds are hence even steadier in this season than in the first season of the year. The mean wind direction is W.N.W.

“The air movement during the remaining five months of the year is essentially of a transitional character. During the second period comprising the months of March, April and May the mean winds in South Madras shift in direction from East in March and April to West in May, in which month, they are practically indetical in direction with the mean winds in June. The mean direction of the winds at Augustia during this period shift *pari passu* with the change of direction of the air movement in Southern India and the resultant is almost nil.

“The conditions in the fourth period including the months October and November are similar to those of the second period except that the transition or change is inverse to that of the second period.

“The number of Calms reported is large; 16% of the wind observations received during this period in 1856-58 were of Calms, so that they were almost as numerous as during the first transitional period March to May. It is also noteworthy that Calms are more numerous in this, as in the second season, during the day than during the night and are most frequent from 11 A.M. to 4 P.M.”

Summary of weather conditions at Augustia.

December, January and February	Calms	5%	}
	East	64%	
	N.E.	21%	
	Other directions	10%	
March, April and May	Calms	19%	}
	East	25%	
	S.W.	33%	
	Other directions	23%	
June to September	Calms	4%	}
	West	72%	
	N.W.	21%	
	Other directions	3%	
October to November	Calms	16%	}
	West	25%	
	East	32%	
	Other directions	27%	

It will be noticed that the time of maximum thunderstorm activity coincides with the time of minimum horizontal air movement at Augustia. *Absence of strong horizontal winds in the higher layers is a sine que non of strong ascending currents favourable to thunderstorm development.*

During the months of March and April, when the general gradient is undefined and is too weak to exercise any control over the air movement, ascending currents begin to rise with the heat of the sun and form detached cumulus heads. These cumuli grow and with the setting in of a light humid breeze from the sea in the afternoon, they grow into large masses often crowned with false cirrus, especially to the north-east of Trivandrum where the land rise is the most marked.

By about 3 P.M. the whole sky is clouded and occasional crashes of thunder are heard. This continues for varying intervals of time and is often accompanied by rain. Ordinarily, the sky clears by about 6 P.M. and a clear night follows. As the season advances, however, the thunderstorm continues at night. The general character of the weather is very similar in October.

These considerations should apply to other places where thunderstorms are seasonal. All along the west coast of India and in Ceylon, the setting in of the monsoon is preceded by a period of thunderstorm activity.

XII. An Optical Study of Free and Forced Convection from Thin Heated Wires in Air.

By Satish Chandra Pramanik, M.Sc., Research Scholar in
the Indian Association for the Cultivation of Science.

(Plates VIII and IX.)

CONTENTS.

- I.—Introduction.
- II.—Application of the Method of Striae.
- III.—Description of the Photographic Records.
- IV.—Deflection of the Rays of light by a Hot wire.
- V.—Summary and Conclusion.

I.—INTRODUCTION.

The problem of convection has received much attention in recent years and an extensive literature has grown up around it, many of the papers dealing with the convective flow of heat from cylinders immersed in fluids—a case which is of importance in view of its practical application in anemometers and gas meters. A very convenient summary of the earlier literature up to the year 1916, is given in a paper by B. B. Ray,¹ which describes also his studies of the form of isothermals in air round a heated cylinder and the optical study of the general nature of convective flow. Several other communications on the subject have also subsequently appeared. K. Aichi² has discussed the problem mathematically following Boussinesq and King, extending their results to the case in which η is not small [$\eta = \frac{vc}{2k}$, v , c and k being respectively the stream velocity, the specific heat and the

¹ Proceedings of the Indian Association for the Cultivation of Science, Vol. VI, Part I, 1920.

² Phys. Math. Soc. Japan Proc. 2, July, 1920.

thermal conductivity of the fluid]. A. H. Davis¹ has also considered the problem for similar bodies theoretically; proceeding from the point of view of the Principle of Similitude, he derived a formula—an extension of Boussinesq's solution to viscous fluids—which he has compared with experimental results of previous workers. He has shown² that "in general both for forced and free convection the hydrodynamical formulae are in very promising agreement with published data except in the case of free convection from hot thin wires." Later on he has shown that provided allowance is made for the temp. change of the properties of the fluid his formulae are substantially satisfactory. On the experimental side Dr. J. S. G. Thomas has published numerous papers.³ His method of investigation is electrical. He has studied both free and forced convection of air (and of also certain other gases) past a hot wire in a cylinder—horizontal and vertical. He has also investigated the influence of one wire on another and on a series of others and considered the bearing of the results thus obtained on anemometry and the construction of gas meters.

The present paper is continuation of Mr. Roy's work, extended to thin wires and is the fruit of the suggestion of Prof. C. V. Raman who has throughout guided its author and rendered him indispensable help.

II.—APPLICATION OF THE METHOD OF "STRIAE."

The experimental arrangement for securing pictures depicting the flow of the air near the hot wire are very similar to those adopted in the Töpler-Schlieren method. An arc lamp *A* (Diagram 1 below) illuminated a circular aperture *H* in a metal screen in front of it and this served as a source of light. By means of a long focus achromatic lens *L* a real image of the illuminated aperture was formed on a thin mica sheet with a circular thin metal knob *K* fixed on to it, which is just sufficient to cover the image. Just behind the mica sheet was placed a telescope *T* focussed on to the middle of the hot wire *W* which lay horizontally along the axis of the optical system.

¹ Phil. Mag. Vol. XL, 1920; Vol. XLV, 1921.

² Phil. Mag. February, 1922.

³ Phil. Mag. May, 1920 and Nov. 1920; Feb. and May, 1921; Feb. and April, 1922, and Proc. Phys. Soc. Vol. XXXII, Part V, 1920

Now, any change of density of the air in the path of a ray would make the ray deviate from its natural course and it would escape the knob and enter the telescope and the image formed by it would give an idea of the disturbance on its path. So any disturbed region on the path of the rays would produce in the telescope a corresponding image. The rays from this image (inverted) after passing through the eye piece would give an erected image on a screen placed behind the telescope. This screen in our experiment, had been the ground glass (and the photographic plate *P* while giving an exposure) of a reflex hand camera *C* from which the lens system had been removed.

The wire was heated electrically, current entering and leaving the wire through copper leads *l, l* passed through capillary glass tubes which allowed the proper fixing up of the wire.

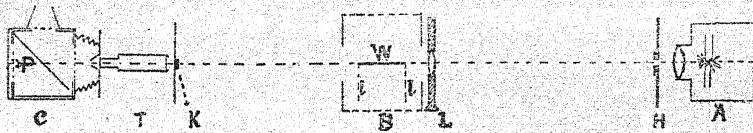


FIG. 17.

To prevent the outside air motion causing any disturbance in regions around the wire, the latter was put inside a wooden box *B* with perforated upper lid, and with partially open sides through which the leads protruded outside so that electrical connections were easy.

In examining the forced convection air was blown horizontally or vertically upwards across the hot wire as desired by means of a fan; to obtain a vertical draught a fan blew air into a funnel which ended in a tube bent up at right angles below the wire. In each case an approximate measure of the velocity of wind was obtained by an ordinary anemometer.

The dimensions of the wire, the values of the heating current and the wind velocity (where used) for the photographs in Plates, VIII and IX are set forth below.

EUREKA WIRES.

I. *Free Convection.*

Fig. No.	Diameter of wires.	Distance between successive wires.	Length along the axis.	Heating currents in amperes.
1	0.054 cm.	—	10 cm.	1.92
2	0.054 "	2.5 cm.	10 "	1.92
3	0.054 "	2.5 "	10 "	1.92
4	0.036 "	0.5 "	7.5 "	1.35
5	0.036 "	0.3 "	7.5 "	1.35
6	0.036 "	0.2 "	7.5 "	1.35
7	0.036 "	0.1 "	7.5 "	1.35
8	0.036 "	0.3 "	7.6 "	1.35

II. *Forced Convection.*

Fig. No.	Diameter of wires.	Distance between successive wires.	Length along the axis.	Heating current in amperes.	Velocity of wind in cm. sec.
9	0.121 cm.	—	20 cm.	6.0	85
10	0.121 "	—	20 "	6.0	117
11	0.036 "	0.3 cm.	7.5 "	1.92	49
12	0.036 "	0.2 "	7.5 "	1.92	49
13	0.036 "	0.2 "	7.5 "	1.35	64
14	0.036 "	0.1 "	7.5 "	1.92	78.5
15	0.036 "	0.3 "	7.6 "	1.35	49
16	0.036 "	0.3 "	7.6 "	1.35	40.5

III.—DESCRIPTION OF THE PHOTOGRAPHIC RECORDS.

In the first figure, the two streams representing the upward flow of hot air by the sides of the wire, are slightly bent towards the right; this is due to the fact that the leads to the wire becoming hot elevates the temp. of the surrounding air which rises up and the adjacent hot stream of air is slightly attracted by it. This phenomenon is more prominent in figures 2 and 3, where the hydrodynamical attraction is very conspicuous. The fourth figure where the arrangement is the same as in figure 2, except that the wires are rather close-together shows how the two middle streams have become shortened. This process continues through the next three figures, the middle streams becoming extinct in the 7th figure where the two wires (separated by a distance of 1 mm.) act as if they were one. In the fifth and the sixth figure there is a remarkable point to be noticed—that the region a little above the wires is cooler than the region higher

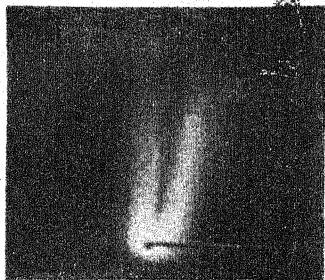


FIG. 1.

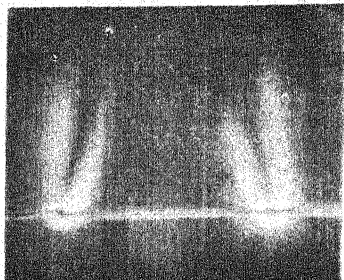


FIG. 2.



FIG. 5.

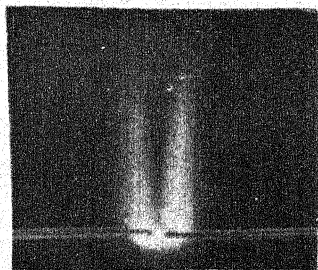


FIG. 6.



FIG. 9.



FIG. 10.



FIG. 13.

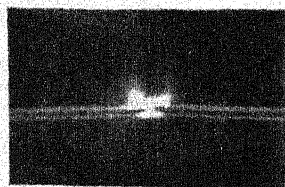
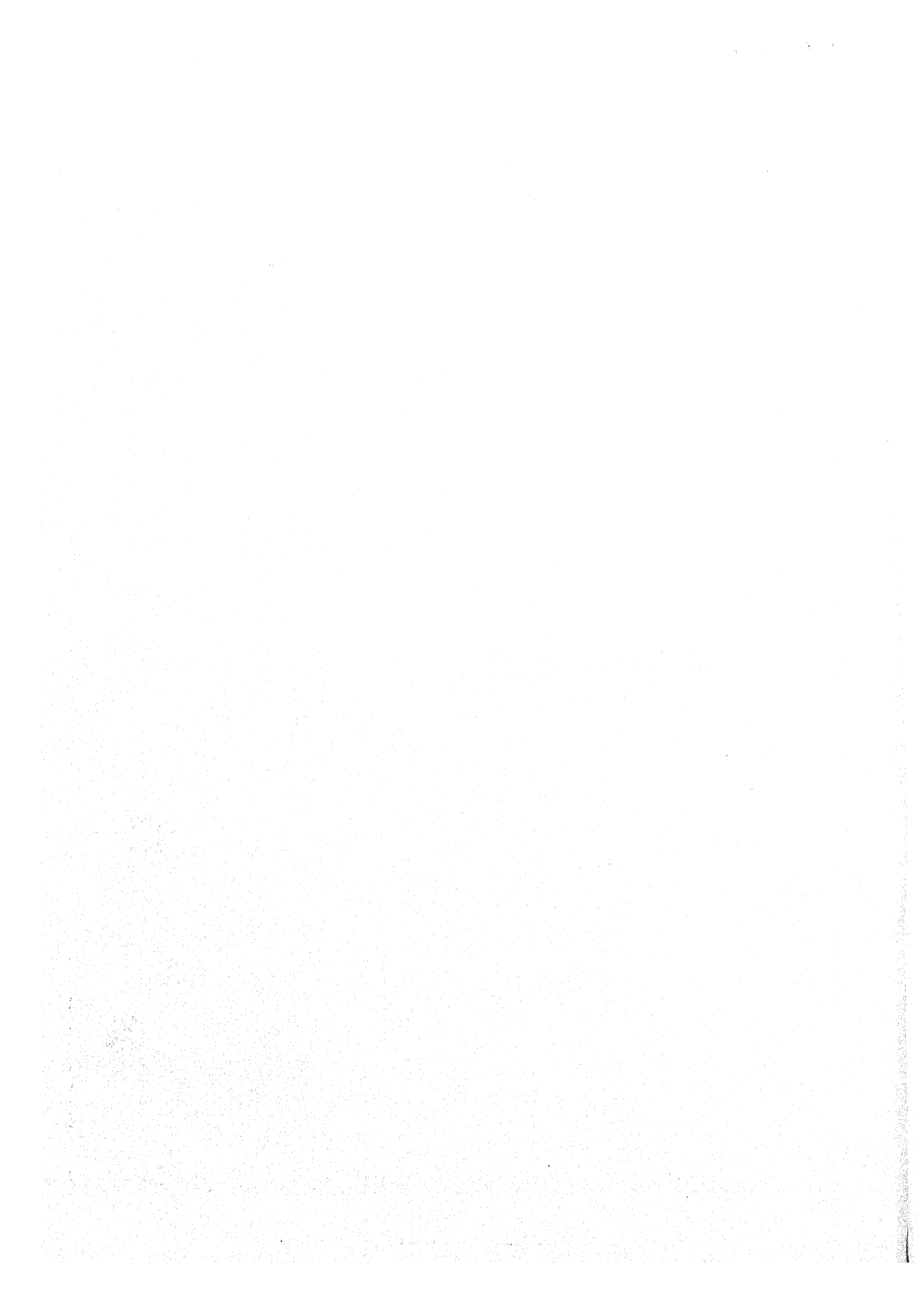


FIG. 14.

Convection from thin heated wires in air.



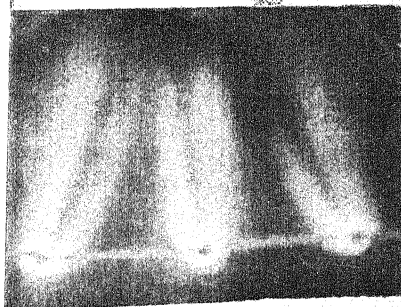


FIG. 3.

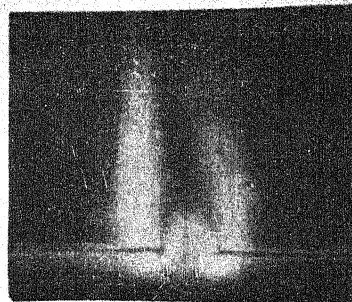


FIG. 4.

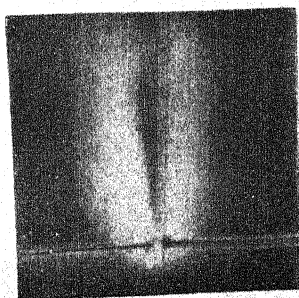


FIG. 7.

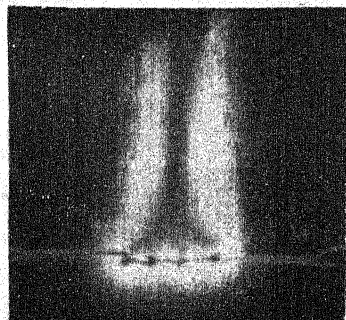


FIG. 8.

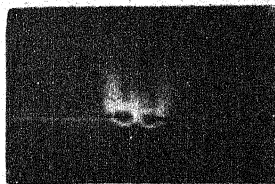


FIG. 11.

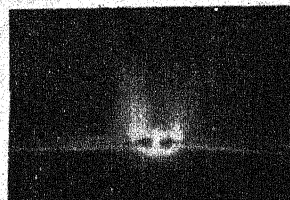


FIG. 12.

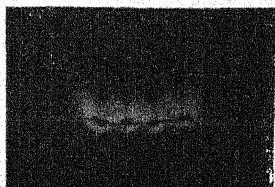


FIG. 15.

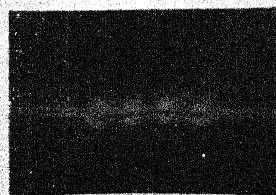


FIG. 15.

Convection from thin heated wires in air.

up. This seems to be due to the fact that the air there remains stationary; it is too far to get much heat by conduction and too near the wires to be disturbed by convected air from surrounding regions. The 8th figure shows that, for a grating of parallel wires, all the intermediate streams get annulled, the external ones having an augmented effect. Here the heat from the wires seems to be conducted, through the surrounding film, to the air below which convects it away to the external streams. The grating more or less shields the air above it from any appreciable convection currents, the more effectively so the nearer the grating elements are. The above statement is true only when the velocity of the convection current is rather small.

The rest of the figures represent forced convection from hot wires. The most important fact to be noticed is that there is always a thin film of hot air surrounding the wire, whether thick or thin. Besides those shown in the plate, the author has observed films in a platinum wire of diameter $\cdot 061$ mm. This is no doubt due to the viscous nature of air. That there exists a thin film of gas round to the hot wire had also to be postulated by Dr. Thomas¹ in connection with his determination of ratio of conductivities of any two gases by the hot wire anemometer. Thus any theory of convective flow which neglects viscosity is likely to be inadequate.

The first two figures of the group now under discussion, represents effect of low and high velocities in the case of single wires, the air streams being horizontal. The next four figures show the effect of superimposed vertical air currents in figures 5 to 7. It will be seen that the air currents tends to make the effect of each wire independent of the others. As heat is convected away more quickly, the temp. falls with a consequent diminution in the size of the streams. The last two figures correspond to the Fig. 8 above: in the first of them the air is blown from down upwards and in the second, from the side horizontally across the grating. In view of what has been said before, they require scarcely any special remark.

Besides the facts thus far studied the author sought the effect of the proximity of a cool parallel wire near the one heated.

¹ J. S. G. Thomas in *Phil. Mag.* Vol. 39, 1920, pp. 532-34.

But except a slight shortening of the streams representing a little cooling, he could optically discover no other change, though, in doing so a steady source of light was used. The eureka wire used in this connection had a diameter of $\cdot 36$ mm. and it was heated by a current of 1.35 amperes.

Dr. Thomas¹ finds, in connection with experiments on forced convection from hot wires in cylindrical tubes that "if the air current be horizontal and the wires are also horizontal but perpendicular to the air current the stability of convection currents is the greatest." But while taking photographs for forced convection the author noticed that stability would be greatest if the wires are subjected to an upward transverse stream of air. The difference is no doubt due to the difference in the surroundings of the wires, in the latter case, the wires being almost in the open.

IV.—DEFLECTION OF THE RAYS OF LIGHT BY A HOT WIRE.

Observation of the distribution of luminosity round the heated wire in Fig. 1, Plate VIII, suggests that in its immediate neighbourhood isothermals are circular cylinders co-axial with it, the deviation from this form as we move away from the wire being much less marked below the wire than above it. The suggestion naturally arises whether it would not be possible to determine the distribution of temperature around the heated wire by a direct optical method.

The first attempt in this direction was made by securing a record of the boundary and the luminosity of the image formed on the ground glass in the camera, by the deviated rays escaping a sharp knife edge placed in place of the knob. The contour lines for the various positions of the knife edge extending from the position where it just cut off the usual image of the aperture to the position where it cut off all the deviated light, were recorded on a piece of translucent paper and they were quite characteristic. But in the absence of a complete mathematical analysis connecting the shape and luminosity of the images with the position of the knife edge and the temperature distribution round the heated wire, the process could not give any tangible results.

Now, whatever might be the actual distribution of temper-

¹ Phil. Mag. 1920, May, pages 519- .

ature round the hot wire, it is certain that the temperature diminishes as we recede from the wire. So a ray incident along the wire must get deviated upwards.¹ Thus it is a priori evident that with incident light parallel and along the wire, we should get a hollow cone of rays (see Diagram 2 below) after they have traversed the disturbed region and if we put a screen just beyond the wire on it should be found a dark spot surrounded by a luminous halo.

But as the region of hot air round the wire is rather small and the temperature gradient is not insignificant, this cone of rays would, at a distance considerable compared with the thickness of the hot region, be intersected by rays that come up after traversing the cooler air distant from the wire. This would limit the minimum width of the dark spot and the boundary after this critical distance

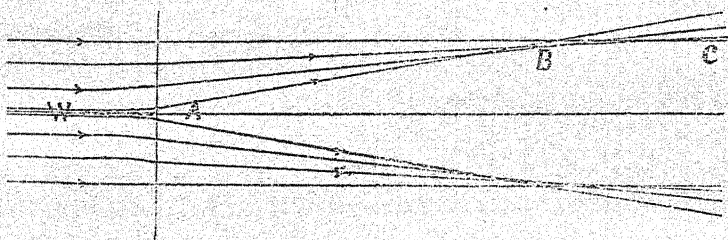


FIG. 18.

would be more and more hazy. Moreover the dark spot would lose its simple circular shape at a distance owing to the asymmetry produced by the two upwards hot streams of air; the effect of these streams (which are cooler than the air very near the wire) would not however be appreciable unless the dark spot is observed at a considerable distance from the end of the hot wire.

In order to test this theoretical conclusion and if possible to apply the experimental results in the determination of temperature, the source of light was brought nearer the lens so that a parallel pencil of light was incident on the disturbed region round a hot platinum wire of diameter $\cdot 0416$ cm. Now, on looking through a microscope focussed near the end of the wire a circular dark spot much larger in size than that due to the obstruction of

¹ Atmospheric refraction of sound due to convective equilibrium of temperature—Rayleigh's Sound, Vol. II, Art. 288.

light by the wire itself was observed. This dark spot, though of diminished size, could be seen even at points round the wire at which the light rays had not passed over its full length. Now as the microscope was gradually moved away from the wire, the spot gradually increased in size but lost its symmetry, the effect of the upward stream of hot air presenting itself. Farther off the spot became faint over its boundary and its size reached a limiting value; the radius of the spot at such a point representing the extent to which the air below the wire gets heated.

For quantitative work, the diameter of the dark spot was measured at various distances from the wire by means of a travelling microscope. From such measurements graphs were drawn of the radius of the dark spot against the distance from the end of the hot wire. The line *A B C* in Diagram 2 indicates, though in an exaggerated way, the general form of the curves obtained, thus confirming in a general way the indications of theory. From these curves and the known values of the effect of temperature on the refractive index of air, coupled with an assumed logarithmic distribution of temp. for the regions below the hot wire, values for the temp. of the air immediately below the wire were found for various magnitudes of heating currents. The values thus obtained, though of the same order of magnitude as those found by resistance measurement were not quite satisfactory. It appears to the author that with a more exact mathematical formulation of the problem the method may prove quite successful.

Some preliminary attempts at measuring the distribution of temperature around the wire by an interference method was also made. It is hoped to repeat these at an early opportunity and to present the results in a separate paper.

V.—SUMMARY AND CONCLUSION.


1. After briefly reviewing recent work on the convection problem, the paper describes the results of an optical study of convective flow around hot thin wires. Three distinct methods of study are available: (a) the method of "striae," (b) the direct optical observations of the deflection of the rays of light in passing over the wire, and (c) the interference method. The results obtained by the first two methods are described.

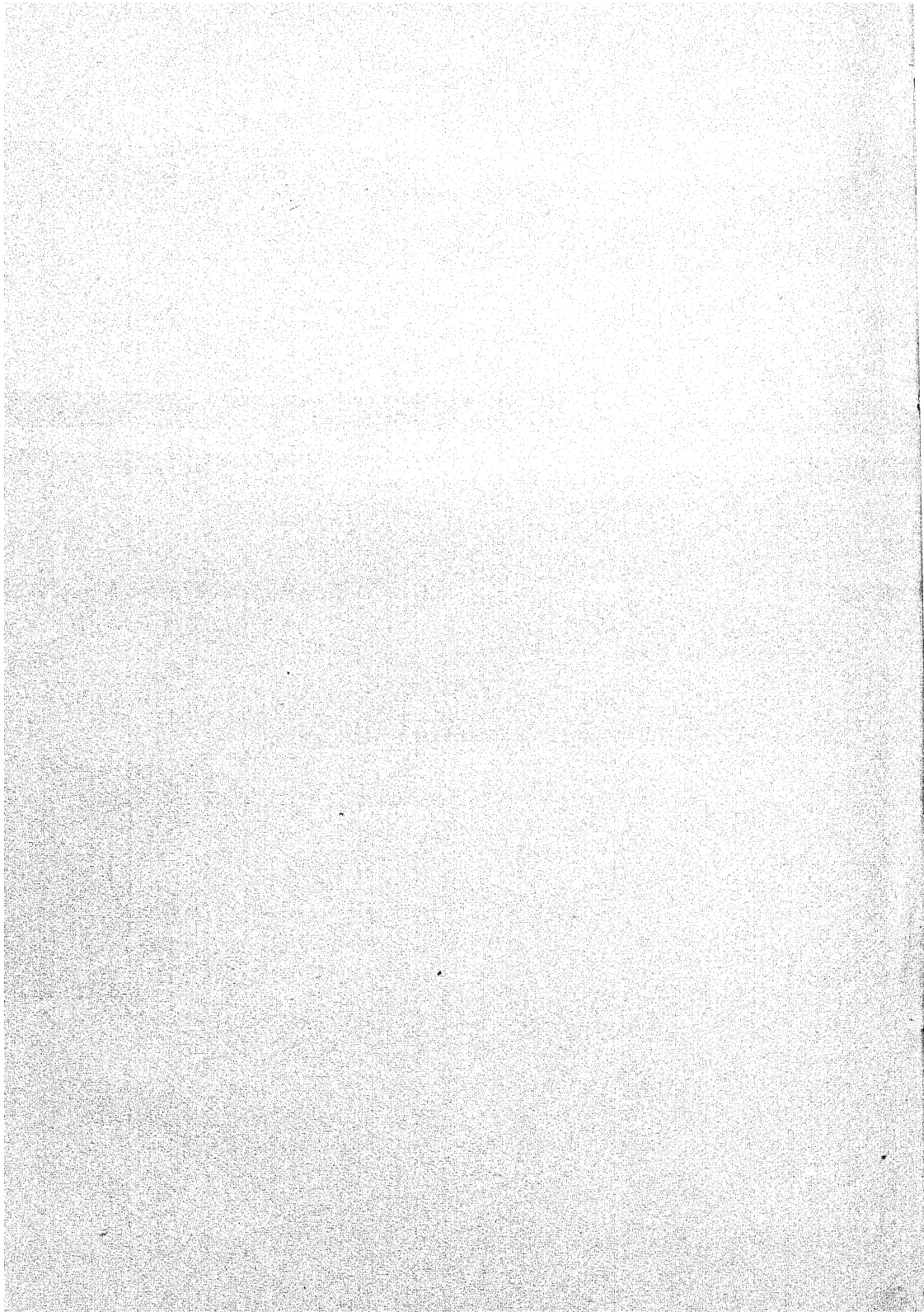
2. The method of "striae" enables an almost exact picture of the distribution of hot air and its temperature variations as indicated by the luminosity of its various parts to be obtained. It is found that as in the case of thick cylindrical wires, so in the case of thin wires, the wire is completely enveloped by a film of hot air both in free and in forced convection; the effect presumably is the result of the viscosity of the gas. In the immediate neighbourhood of the wire the flow of heat is mainly by conduction and in the more distant parts by convection. The mutual effect of convection-currents arising from parallel heated wires, and especially the remarkable impedance offered by a parallel grating of wires for the flow of hot air through it in free convection and to a lesser extent in forced convection is illustrated by photographs.

3. These results are supported by the observations by the second method. The possibility of obtaining quantitative results is also considered.

In conclusion, my best thanks are due to Dr. C. V. Raman, Palit Professor of Physics, University College of Science, who has throughout rendered the author invaluable help and put the facilities of a well-equipped laboratory at his disposal.

*Indian Association for the Cultivation of Science,
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5th May, 1922.*





XIII. On Laminar Diffraction and the Theory of Microscopic Vision.

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SYNOPSIS.

1. *Theory of Microscopic vision.*—The mathematical theory as developed by Rayleigh, Lummer and Reiche, and Wolfke is applied in the present paper to determine the character of the images of different types of *transparent* structures under the microscope.
2. *A Discontinuous Laminar Boundary.*—The image of the boundary is shown to be a place of minimum intensity bordered by fine *equidistant* fringes on either side. The position of the fringes is sensibly independent of path retardation on the two sides of the boundary.
3. *An Echelon Boundary of Two Steps.*—The images of the two boundaries are places of minimum intensity compared with the other fringes bordering them. For steps of a given width, the position of maxima and minima of illumination in the field is sensibly independent of the height of the steps. The nature of the microscopic image, however, depends on the *separation* of the steps, and the diminishing clearness of resolution with their closer approach, is illustrated by three graphs.
4. *A Laminar Transparent Ridge.*—In this case, as the ridge becomes narrower and narrower, besides the diminution in the clearness of resolution of the two edges, the entire image tends to disappear, being replaced by uniform illumination.
5. *Experimental verification of the Results.*—The results indi-

cated by theory have been tested by observations under the microscope of various actual structures: (1) the striae in mica; (2) air bubbles in a thin film of liquid between two glass plates; (3) fine ridges of egg-albumen formed by drying up of a 'mixed' plate; (4) fungus growths on glass; and (5) botanical and petrographical slides of various kinds. While the results of theory are broadly speaking supported by observation, a certain asymmetry in the appearance of laminar boundaries is generally observed, which is not explained by the theory worked out in the paper. An explanation of the same is suggested, which, however, requires further investigation.

6. *Effect of Oblique Illumination*.—The asymmetry referred to in (5) becomes greatly magnified under oblique illumination.
7. *Rayleigh's Theory of the Phenomena Observed in Foucault's Test*.—The close similarity between this and the theory of microscopic vision is pointed out.

I.—INTRODUCTION.

The importance of the part played by diffraction in determining or modifying the character of the images of minute structures as seen through a microscope, has been widely recognised through Abbe's well-known work on the nature of microscopic images.¹ Several later writers have also dealt with the subject, notably the late Lord Rayleigh, Porter, Wolfe and others.² The published literature deals, however, mostly with the special case in which the object under observation is a grating formed of alternate transparent and opaque bars, and very little has been written about other cases, such as for instance that of laminar gratings of complex groove-form, the visibility of whose structure depends on the periodic variations of phase of the light passing through it and not upon a variation of opacity. So far as the

¹ Abbe—*Die Lehre von der Bildenstehung im Mikroskop*. Bearbeitet und herausgegeben von O. Lummer and F. Reiche. (Fr. Vieweg & Sohn, Braunschweig 1910, 108 pp.)

² For references to the literature, see the paper by P. N. Ghosh. *Phy. Rev.* Dec. 1919.

writer is aware, even the problem of the nature of the focussed image under the microscope of a simple laminar boundary, or of two or more parallel laminar boundaries has not been mathematically discussed. Cases of the kind above mentioned are of special interest in that the microscopic images of such structures are entirely spurious in the sense that the observed fluctuations of intensity in the field of view do not represent any actual fluctuations in the intensity of the light passing through the object. The consideration of such cases is probably quite as important as that of the simple black and white grating in interpreting the observed appearances under the microscope of various natural structures. It was with a view to fill up the gap in the literature here indicated that the writer undertook the investigation, the first instalment of the results of which are contained in the present paper. Some experimental illustrations of the mathematical theory of these cases have also been worked out.

II.—GENERAL STATEMENT OF THEORY.

In their treatise on Abbe's theory, Lummer and Reiche¹ have given an expression for the amplitude of light wave at a point in the plane conjugate to the object-plane of the system. The disturbance S_2 at any point of observations in the image-plane conjugate to the point (x, y) in the object-plane is given by

$$S_2 = \frac{k}{\lambda^2} \iint_{\text{aperture}} d\xi' d\eta' \iint_{\text{object}} dXdY \phi(XY) \sin 2\pi \left[\frac{t}{T} - \psi(XY) - \frac{\xi'(x-X)}{\lambda} - \frac{\eta'(y-Y)}{\lambda} \right] \dots\dots (1)$$

where k is a constant, λ the wave length of light, $\phi(XY)$ the transmission coefficient of the object, and $\psi(XY)$, the phase-

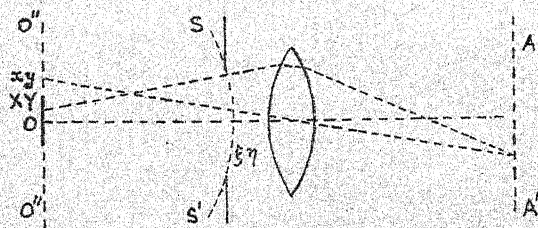


FIG. 1.

¹ Loc. cit.

change due to the object, and (ξ, η) are the coordinates of an element of ss' a sphere of radius e , described round the mid-point o of the object as shown in figure 1.

$$\xi' = \frac{\xi}{e} \text{ and } \eta' = \frac{\eta}{e}$$

are the angular height and breadth of the element. The object is illuminated by plane waves incident normally on it, so that all the diffracted rays originally start from it in the same phase. The dimensions of the object are small compared with e .

For a symmetrical object extending between the limits $-A$ and $+A$ on the x -axis and $-B$ and $+B$ on the y -axis, Wolfke¹ transforms the equation (1) to

$$S_2 = \frac{k}{\lambda} \int_{-a}^{+a} d\xi' \int_{-A}^{+A} dX \phi(X) \sin 2\pi \left[\frac{t}{\tau} - \frac{\xi'(x-X)}{\lambda} \right] \dots\dots\dots (2)$$

the object being supposed to introduce no phase-change, i.e. $\psi(XY)=0$, and a' is the angular height of the aperture of the objective, it being supposed to be rectangular. For a transparent object, we may $\phi(X)=1$. Further putting

$$\frac{x}{e} = \phi, \frac{X}{e} = \theta, \xi' = \frac{\xi}{e}, \text{ and } \pm ea' = \pm \xi,$$

the equation (2) transforms to

$$S_2 = \frac{k}{\lambda} \int_{-\xi}^{+\xi} d\xi \int_{-\theta}^{+\theta} d\theta \sin 2\pi \left[\frac{t}{\tau} - \frac{\xi(\phi-\theta)}{\lambda} \right]$$

or

$$S_2 = \frac{k}{\lambda} \int_{-\xi}^{+\xi} d\xi \int_{-\theta}^{+\theta} d\theta \sin \frac{2\pi}{\lambda} \left[Vt - \xi(\phi-\theta) \right] \dots\dots\dots (3)$$

where $\pm \xi$ are the limits of the aperture, and $\pm \theta$ are the angular limits of the object. Omitting the constant factor $\frac{k}{\lambda}$ for brevity, and putting $Vt=\tau$ and $\lambda=2\pi$, which can always, be restored on considering 'dimensions,' we have

$$S_2 = \int_{-\xi}^{+\xi} d\xi \int_{-\theta}^{+\theta} d\theta \sin \left\{ \tau - \xi(\phi-\theta) \right\} \dots\dots\dots (4)$$

¹ Wolfke—Annalen der Physik, 34, 1911.

For a transparent object introducing a constant retardation ρ , equation (2) transforms to

$$S_2 = \int_{-\xi}^{+\xi} d\xi \int_{-\theta}^{+\theta} d\theta \sin \left\{ \tau - \rho + \xi(\theta - \phi) \right\} \dots\dots\dots (5)$$

we now proceed to apply equation (5) to different types of laminar structures.

III.—CASE OF A DISCONTINUOUS LAMINAR BOUNDARY.

Let us first take the case of a transparent object, having a discontinuous laminar boundary. This, we may suppose, introduces an otherwise constant retardation ρ , which changes sign when $\theta=0$. The retardation is equal to $+\rho$, when θ is positive, and is equal to $-\rho$, when θ is negative

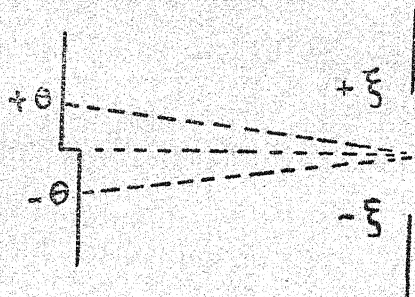


FIG. 2.

In this case equation (5) transforms to

$$S_2 = \int_{-\xi}^{+\xi} d\xi \left[\int_{-\theta}^0 d\theta \sin \left\{ (\tau + \rho - \xi\phi) + \xi\theta \right\} + \int_0^{\theta} d\theta \sin \left\{ (\tau - \rho - \xi\phi) + \xi\theta \right\} \right]$$

Performing the integration with respect to θ , we have

$$S_2 = \int_{-\xi}^{+\xi} d\xi \left[\frac{\sin \xi\theta}{\xi} \left\{ \sin (\tau + \rho - \xi\phi) + \sin (\tau - \rho - \xi\phi) \right\} - \frac{(1 - \cos \xi\theta)}{\xi} \left\{ \cos (\tau + \rho - \xi\phi) - \cos (\tau - \rho - \xi\phi) \right\} \right]$$

$$\begin{aligned}
&= 2 \sin \tau \left[\int_{-\xi}^{+\xi} d\xi \left\{ \cos \rho \cdot \frac{\sin \xi \theta \cos \xi \phi}{\xi} + \sin \rho \cdot \frac{(1 - \cos \xi \theta)}{\xi} \cos \xi \phi \right\} \right] \\
&- 2 \cos \tau \left[\int_{-\xi}^{+\xi} d\xi \left\{ \cos \rho \frac{\sin \xi \theta \sin \xi \phi}{\xi} + \sin \rho \frac{(1 - \cos \xi \theta)}{\xi} \sin \xi \phi \right\} \right] \\
&= \sin \tau \cos \rho \cdot \int_{-\xi}^{+\xi} d\xi \frac{\sin (\theta + \phi) \xi + \sin (\theta - \phi) \xi}{\xi} + \sin \tau \sin \rho \int_{-\xi}^{+\xi} d\xi \frac{\cos \xi \phi}{\xi} \\
&- \sin \tau \sin \rho \int_{-\xi}^{+\xi} d\xi \frac{\cos (\theta + \phi) \xi + \cos (\theta - \phi) \xi}{\xi} \\
&- \cos \tau \cos \rho \int_{-\xi}^{+\xi} d\xi \frac{\cos (\theta - \phi) \xi - \cos (\theta + \phi) \xi}{\xi} \\
&- \cos \tau \sin \rho \left[\int_{-\xi}^{+\xi} d\xi \frac{2 \sin (\phi \xi) - \sin (\theta + \phi) \xi + \sin (\theta - \phi) \xi}{\xi} \right] \dots \dots (6)
\end{aligned}$$

The integrals in (6) may be expressed in terms of sine-integral and cosine-integral defined by

$$\text{si}(x) = \int_0^x \frac{\sin x}{x} dx$$

and

$$\text{ci}(x) = \int_{\infty}^x \frac{\cos x}{x} dx.$$

As the aperture of the objective of the microscope is symmetrical with respect to the axis, the ci's being even functions disappear, and (6) reduces to

$$\begin{aligned}
S_2 &= 2 \sin \tau \cos \rho \cdot \int_0^{\xi} d\xi \frac{\sin (\theta + \phi) \xi + \sin (\theta - \phi) \xi}{\xi} \\
&- 2 \cos \tau \sin \rho \cdot \int_0^{\xi} d\xi \frac{2 \sin (\phi \xi) - \sin (\theta + \phi) \xi + \sin (\theta - \phi) \xi}{\xi} \\
&= 2 \sin \tau \cos \rho \cdot \{ \text{si} (\theta + \phi) \xi + \text{si} (\theta - \phi) \xi \} \\
&- 2 \cos \tau \sin \rho \cdot \{ 2 \text{si} (\phi \xi) - \text{si} (\theta + \phi) \xi + \text{si} (\theta - \phi) \xi \}
\end{aligned}$$

Hence the intensity I is given by

$$I = 4 \cos^2 \rho \cdot \{ \text{si} (\theta + \phi) \xi + \text{si} (\theta - \phi) \xi \}^2 \\ + 4 \sin^2 \rho \cdot \{ 2 \text{si} (\phi \xi) - \text{si} (\theta + \phi) \xi + \text{si} (\theta - \phi) \xi \}^2.$$

The intensity at different directions ϕ , for a definite value of θ , may be easily computed by the aid of the tables calculated by Glaisher.¹

In some actual observations to be described later,

$$\lambda = 5 \times 10^{-5} \text{ cm. } 2\xi = .20 \text{ cm. } \theta = \frac{1}{21}$$

approximately. Hence $\frac{2\pi}{\lambda} \theta \xi = 598$. For facility in calculation

we may take $\theta \xi$ or rather $\frac{2\pi}{\lambda} \theta \xi = 600$ approximately. We now proceed to tabulate the values of intensity for different values of the retardation ρ . When the whole difference of retardation $2\rho = \pi$, or reckoned in wave lengths $= \frac{\lambda}{2}$, we have $\cos \rho = 0$ and $\sin \rho = 1$.

It may be noted that $\theta \xi$ being 600, and the variations in the values of the si-function in the neighbourhood of $\text{si} (600)$ being very small, the expression for the intensity for values of $\phi \xi$ small compared with $\theta \xi$, reduces to

$$I = 16 \{ \text{si} (\phi \xi) \}^2 \dots\dots\dots (8)$$

The fluctuations of intensity with $\frac{\phi}{\theta}$ or more conveniently with $\frac{2\pi}{\lambda} \phi \xi$ is exhibited in Fig. 3.

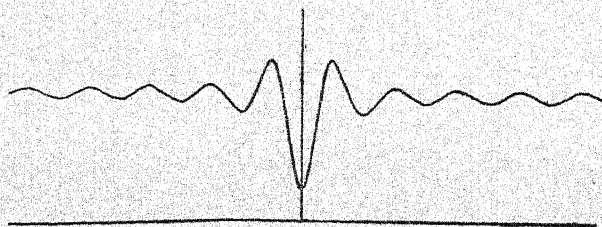


FIG. 3.

The intensity is symmetrical with respect to $\phi \xi = 0$, and the fringes on either side of the central black line are equidistant.

¹ J. W. L. Glaisher, Phil. Trans. 1870.

Similarly for any other value of 2ρ , the variations in intensity may be calculated from (7). Only one other graph illustrating the case of $2\rho = \frac{\lambda}{3}$ has been shown.

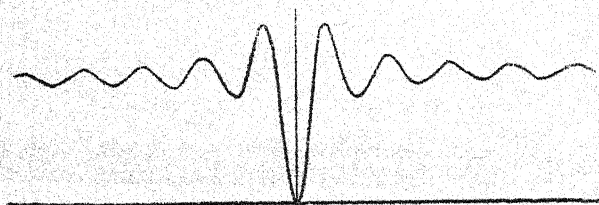


FIG. 4.

We therefore conclude that the geometrical image of the laminar boundary is always a place of minimum intensity, compared with other minimas bordering it on either side, and the positions of maxima and minima are the same for any value of ρ , except when $2\rho = \lambda$, which leads to uniform illumination in the field of view of the microscope.

IV.—CASE OF AN ECHELON-BOUNDARY OF TWO STEPS.

Next we proceed to consider the case of a transparent object with two stepped-laminar boundaries close together as shown in Fig. 5.

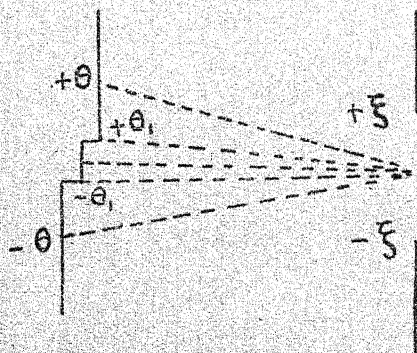


FIG. 5.

The angle subtended by the boundaries at the mid-point of the aperture of the objective is $2\theta_1$. In this case, we may suppose the retardation to be constant and $= -\rho$ from $-\theta$ to $-\theta_1$,

and $+\rho$ from $+\theta_1$ to $+\theta$, and to be zero from $-\theta_1$ to $+\theta_1$, so that each step introduces a retardation of ρ_1 and the whole difference of retardation produced by the two boundaries is 2ρ .

Equation (5) in this case transforms to

$$\begin{aligned}
 S_2 &= \int_{-\xi}^{+\xi} d\xi \left[\int_{-\theta}^{-\theta_1} d\theta \sin \left\{ \tau + \rho + \xi (\theta - \phi) \right\} + \int_{-\theta_1}^{+\theta_1} d\theta \sin \left\{ \tau + \xi (\theta - \phi) \right\} \right. \\
 &\quad \left. + \int_{\theta_1}^{\theta} d\theta \sin \left\{ \tau - \rho + \xi (\theta - \phi) \right\} \right] \\
 &= \int_{-\xi}^{+\xi} d\xi \left[\frac{(\sin \xi\theta - \sin \xi\theta_1)}{\xi} \left\{ \sin (\tau + \rho - \xi\phi) + \sin (\tau - \rho - \xi\phi) \right\} \right. \\
 &\quad \left. + 2 \sin (\tau - \xi\phi) \frac{\sin \xi\theta_1}{\xi} - \frac{(\cos \xi\theta - \cos \xi\theta_1)}{\xi} \left\{ \cos (\tau - \rho - \xi\phi) \right. \right. \\
 &\quad \left. \left. - \cos (\tau + \rho - \xi\phi) \right\} \right] \\
 &= \int_{-\xi}^{+\xi} d\xi \left[\frac{(\sin \xi\theta - \sin \xi\theta_1)}{\xi} (2 \sin \tau \cos \rho \cos \xi\phi - 2 \cos \tau \cos \rho \sin \xi\phi) \right. \\
 &\quad \left. + \frac{2 \sin \tau \cos \xi\phi \sin \xi\theta_1}{\xi} - \frac{2 \cos \tau \sin \xi\phi \sin \xi\theta_1}{\xi} \right. \\
 &\quad \left. - \frac{(\cos \xi\theta - \cos \xi\theta_1)}{\xi} (2 \sin \tau \sin \rho \cos \xi\phi - 2 \cos \tau \sin \rho \sin \xi\phi) \right]
 \end{aligned}$$

Omitting those terms which vanish on integration between the limits $\pm\xi$, we have

$$\begin{aligned}
 S_2 &= 2 \sin \tau \cos \rho \int_{-\xi}^{+\xi} d\xi \cdot \frac{\sin \xi\theta \cos \xi\phi}{\xi} - 2 \sin \tau \cos \rho \cdot \int_{-\xi}^{+\xi} d\xi \frac{\sin \xi\theta_1 \cos \xi\phi}{\xi} \\
 &\quad + 2 \sin \tau \int_{-\xi}^{+\xi} d\xi \frac{\cos \xi\phi \sin \xi\theta_1}{\xi} + 2 \cos \tau \sin \rho \cdot \int_{-\xi}^{+\xi} d\xi \frac{\cos \xi\theta \sin \xi\phi}{\xi} \\
 &\quad - 2 \cos \tau \sin \rho \int_{-\xi}^{+\xi} d\xi \frac{\cos \xi\theta_1 \sin \xi\phi}{\xi} \\
 &= 2 \sin \tau \left[\cos \rho \left\{ \text{si } (\theta + \phi)\xi + \text{si } (\theta - \phi)\xi - \text{si } (\theta_1 + \phi)\xi - \text{si } (\theta_1 - \phi)\xi \right\} \right. \\
 &\quad \left. + \text{si } (\theta_1 + \phi)\xi + \text{si } (\theta_1 - \phi)\xi \right] \\
 &\quad + 2 \cos \tau \left[\sin \rho \cdot \left\{ \text{si } (\theta + \phi)\xi - \text{si } (\theta - \phi)\xi - \text{si } (\theta_1 + \phi)\xi + \text{si } (\theta_1 - \phi)\xi \right\} \right]
 \end{aligned}$$

Let A and B denote the coefficients of $\sin \tau$ and $\cos \tau$ respectively. Then

$$I = A^2 + B^2.$$

We shall now determine the nature of the diffraction pattern for $\cos \rho = 0$, i.e. when $2\rho = \frac{\lambda}{2}$ for different values of θ, ξ . The cases which present interesting features are when the two boundaries are close together, and we shall calculate the intensity pattern for different degrees of closeness.

For $\cos \rho = 0$

$$I = 4 \left[\{ \text{si}(\theta_1 + \phi)\xi + \text{si}(\theta_1 - \phi)\xi \}^2 + \{ \text{si}(\theta_1 - \phi)\xi - \text{si}(\theta_1 + \phi)\xi \}^2 \right] \\ = 8 \cdot \left[\{ \text{si}(\theta_1 + \phi)\xi \}^2 + \{ \text{si}(\theta_1 - \phi)\xi \}^2 \right]$$

As before $\frac{2\pi}{\lambda} \theta_1 \xi = 600$ approximately. First let $\frac{2\pi}{\lambda} \theta_1 \xi = 6$. The diffraction pattern in this case is shown in Fig. 6.

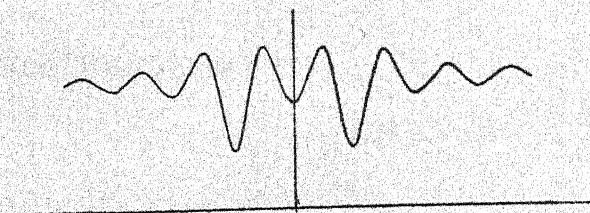


FIG. 6.

Next let $\frac{2\pi}{\lambda} \theta_1 \xi = 4$. The steps are closer than in the preceding case. Plotting the values of intensity for different values of $\phi \xi$ we get Fig. 7.

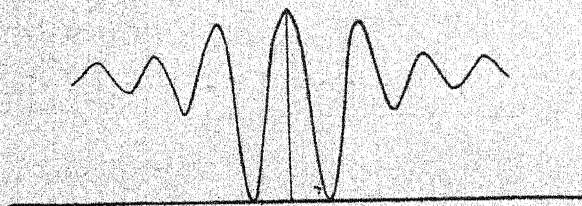


FIG. 7.

From Fig. 6 we see that the laminar borders appear as places of minimum intensity as compared with other minimas bordering them. Between them we have some slight variations of intensity, as is shown by the presence of a dark fringe at the centre, and a bright one accompanying it. From Fig. 7 we find that the images of the laminar borders are again regions of minimum intensity,

but in this case the intensity is greatest at the centre, and gradually decreases from centre to the images on either side. As we pass outwards, we meet with fluctuations of intensity gradually leading to uniform illumination. Also we conclude from the figures that the nature of the diffraction pattern between the laminar boundaries depends on the separation of the steps.

Next let $\theta_1\xi=1.6$. Here the two laminar boundaries are just resolved.

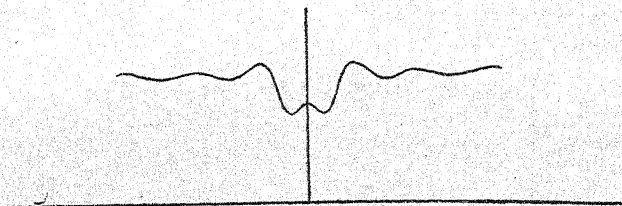


FIG. 8.

Fig. 8 shows that the two boundaries are places of minimum intensity, with a brighter region between them. On either side there is the usual fluctuation of intensity rapidly ending in uniform illumination.

Lastly let $\theta_1\xi=1$. As the diffraction pattern in this case is very similar to that of Fig. 4, it has not been shown. In this case we get a region of minimum intensity along $\phi\xi=0$, and the intensity gradually increases in either direction, till we get a region of maximum intensity. The laminar boundaries cannot be located in this case; in other words, they are too close to be resolved by the microscope.

Figs 6, 7, and 8 have been drawn for the case when $2\rho=\frac{\lambda}{2}$, but actual calculations for values of $\theta\xi=600$, θ , $\xi=4$, and for $\cos \rho=\frac{1}{2}$ and $\cos \rho=1$, i.e. when the total path-retardation $2\rho=\frac{\lambda}{3}$ and λ respectively, show that the positions of maxima and minima agree quite closely with those shown in Fig. 7.

Hence we may conclude as in the first case, that for definite values of $\theta\xi$ and $\theta_1\xi$, the positions of maxima and minima are the same for any value of the path retardation, though the numerical values of intensity at the corresponding points may be different.

V.—CASE OF A TRANSPARENT RIDGE.

Lastly, we proceed to consider the case of a transparent object, with two close ridges as shown in Fig. 9.

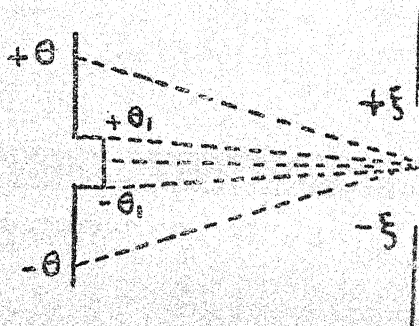


FIG. 9.

The angle subtended by the two ridges at o , the mid-point of the aperture of the objective is $2\theta_1$. In this particular case, the linear retardation may be taken to be constant and equal to $-\rho$ from $-\theta$ to $-\theta_1$, and $+\rho$ from $-\theta_1$ to $+\theta_1$, and again $-\rho$ from θ_1 to θ . Equation (5) then transforms to

$$\begin{aligned}
 S_2 &= \int_{-\xi}^{+\xi} d\xi \left[\int_{-\theta}^{-\theta_1} d\theta \sin \{ \tau + \rho + \xi(\theta - \phi) \} + \int_{-\theta_1}^{+\theta_1} d\theta \sin \{ \tau - \rho + \xi(\theta - \phi) \} \right. \\
 &\quad \left. + \int_{\theta_1}^{\theta} d\theta \sin \{ \tau + \rho + \xi(\theta - \phi) \} \right] \\
 &= \int_{-\xi}^{+\xi} d\xi \left[\sin(\tau + \rho - \xi\phi) \frac{(\sin \xi\theta - \sin \xi\theta_1)}{\xi} + \frac{\sin \xi\theta_1}{\xi} \sin(\tau - \rho - \xi\phi) \right] \\
 &= 2 \sin(\tau + \rho) \int_{-\xi}^{+\xi} d\xi \frac{\cos \xi\phi \sin \xi\theta}{\xi} - 2 \cos(\tau + \rho) \int_{-\xi}^{+\xi} d\xi \frac{\sin \xi\phi \sin \xi\theta}{\xi} \\
 &\quad - 2 \sin(\tau + \rho) \int_{-\xi}^{+\xi} d\xi \frac{\cos \xi\phi \sin \xi\theta_1}{\xi} + 2 \cos(\tau + \rho) \int_{-\xi}^{+\xi} d\xi \frac{\sin \xi\phi \sin \xi\theta_1}{\xi} \\
 &\quad + 2 \sin(\tau - \rho) \int_{-\xi}^{+\xi} d\xi \frac{\cos \xi\phi \sin \xi\theta_1}{\xi} - 2 \cos(\tau - \rho) \int_{-\xi}^{+\xi} d\xi \frac{\sin \xi\phi \sin \xi\theta_1}{\xi}
 \end{aligned}$$

Performing the integration and simplifying we have

$$S_2 = 2 \sin \tau \cos \rho \{ \text{si} (\theta + \phi) \xi + \text{si} (\theta - \phi) \xi \} + 2 \cos \tau \sin \rho \cdot \{ \text{si} (\theta + \phi) \xi + \text{si} (\theta - \phi) \xi - 2 \text{si} (\theta_1 + \phi) \xi - 2 \text{si} (\theta_1 - \phi) \xi \}$$

Let A and B denote the coefficients of $\sin \tau$ and $\cos \tau$ respectively. Then we have

$$I = A^2 + B^2.$$

when

$$\cos \rho = 0, \text{ i.e. } 2\rho = \frac{\lambda}{2},$$

$$I = 4 \{ \text{si} (\theta + \phi) \xi + \text{si} (\theta - \phi) \xi - 2 \text{si} (\theta_1 + \phi) \xi - 2 \text{si} (\theta_1 - \phi) \xi \}^2$$

when $\theta \xi$ is large compared with $\phi \xi$, the expression for the intensity reduces to

$$I = 16 \{ \text{si} \theta \xi - \text{si} (\theta_1 + \phi) \xi - \text{si} (\theta_1 - \phi) \xi \}^2$$

As in the two foregoing cases, let $\frac{2\pi}{\lambda} \theta \xi = 600$ approximately. Figures 10 and 11 have been drawn for $\frac{2\pi}{\lambda} \theta_1 \xi = 5$ and 4 respectively.



FIG. 10.

Fig. 10 illustrates the case for $\theta_1 \xi = 5$. In this case two maxima and two minima appear between the two edges, but the latter can be distinguished by the fact that they are regions of minimum

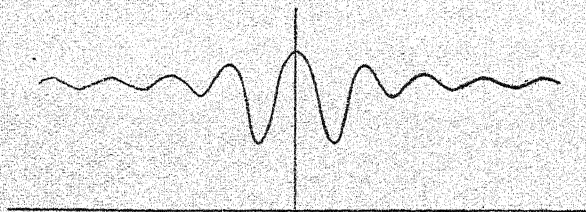


FIG. 11.

intensity. Outside the edges, we have as usual, fluctuations of intensity, gradually ending in uniform illumination. The maxima and minima in the region between the images are feeble in

character. Fig. 11 shows that in this case ($\theta_1\xi=4$), the edges are again regions of minimum intensity. But between these two regions the intensity is greatest, as distinguished from the previous case. Outside the geometrical images of the edges, we have several maxima and minima of illumination.

From these two cases, we also conclude that the diffraction pattern in the region between the images of the edges, depends on the degree of separation between them.

When $\theta_1\xi$ is small, say $=.5$, the diffraction pattern is very similar in appearance to that represented in Fig. 4. In the neighbourhood of $\phi\xi=0$, i.e. along the axis, the intensity is very small; it then gradually increases in either direction, the field of view being symmetrical, till we find a very bright region, accompanied by several maxima and minima of illumination. It is evident that in this case, there is nothing to indicate the geometrical images of the boundaries, i.e. they are so close together, that the microscope fails to separate them.

When the two boundaries are very close together, i.e. $\theta_1\xi$ is very small (less than .1), we find from the above expression for intensity, that it is practically constant throughout the whole field of view, since the 'si' is an odd function. In this case no image of the ridge is seen in the microscope.

The previous figures have been drawn for the case, when the retardation is $2\rho=\frac{\lambda}{2}$ but calculation shows that for fixed values of $\theta\xi$ and $\theta_1\xi_1$ but for different values of $2\rho_1$ except when $2\rho=\lambda$, the diffraction pattern is similar to the corresponding cases of $2\rho=\frac{\lambda}{2}$, and the positions of maxima and minima are the same, though the absolute values of the intensity may be different.

This constant position of maxima and minima in all the three cases discussed above, furnishes us with an easy method of testing our theory experimentally.

VI.—EXPERIMENTAL TESTS OF THE THEORY.

For experimental verification, several types of structure were studied. It is well known that structures occurring in nature, e.g. the cell walls and nuclei-borders of cladophora and the edges of diatoms appear as sharp black lines under the microscope. Under

direct illumination the edges can also be seen surrounded by very fine fringes in the position of accurate focus. Similar instances occur in the case of mixed plates formed of air bubbles and albumen or water.¹ For quantitative measurements, the striae in mica were used. It has been shown by P. N. Ghosh² that a stria is the boundary separating regions in the mica having slightly different thicknesses. When examined by a microscope under direct illumination, the striae appear resolved into five dark lines separated by bright regions. All the edges do not always appear equally dark, which points to the fact, that the successive steps represent unequal changes of thickness in the mica. Further the steps generally are of unequal width, and the width may vary from point to point along the length of a stria.

For the verification of the first case, a stria was chosen which showed a single edge, not resolved by a very high power Leitz objective. The source of light was a pinhole, or a slit parallel to the direction of the stria, illuminated by a powerful electric lamp, and at a sufficient distance from it, so that the mica was illuminated by plane waves incident normally on it. The edge appeared as a fine black line, accompanied by practically equidistant fringes on either side. The preceding theory shows that it is not necessary to determine the path retardation on the two sides of the stria. The microscope was fitted with a micrometer eyepiece reading up

to $\frac{1}{1000}$ cm. The field of view of the mica was limited by two

Gillete razor blades fixed to it, thus forming a slit with the stria in the middle, and its edges parallel to it. It should be noted that this limitation of the field of view of mica is not sufficient to limit the aperture of the objective of the microscope, the full aperture being always effective. This was tested by the fact that the appearance of the stria, and its accompanying fringes was not changed by removing the razor blades. Thus θ could be determined, if necessary. The distance of the focal plane of the microscope from its aperture was $D=30.0$ cms. and its width $2\xi=.201$ cm.

From equation (8) we see that I is a minimum when

¹ C. V. Raman and B. N. Banerjee, *Phil. Mag.*, March 1921.

² P. N. Ghosh, *Proc. Roy. Soc. A*, Vol. 96, 1919.

si $\left(\frac{2\pi}{\lambda} \phi \xi\right)$ is a minimum. From the nature of the si-function we therefore have for the first minimum value of intensity

$$\frac{2\pi}{\lambda} \phi \xi = 2\pi \text{ or } \phi = \frac{\lambda}{\xi},$$

and the successive minima are situated at intervals of 2π .

Taking $\lambda = 5 \times 10^{-5}$ cm. approximately, as corresponding to the brightest part of the visible spectrum, we have d , the distance of the first minimum on either side from the darkest central minimum, given by

$$d = \phi D = \frac{\lambda}{\xi} D = \frac{5 \times 10^{-5} \times 30.0}{.10} = .0151 \text{ cm.}$$

As stated previously $\frac{2\pi}{\lambda} \theta \xi$ is not required for calculation in this particular case. The following table shows the agreement to a close approximation between the observed and calculated values.

$$\gamma = 5 \times 10^{-5} \text{ cm. } D = 30.0 \text{ cm. } 2\xi = .201 \text{ cm.}$$

Serial No.	1	2	3	
Observed distance between successive minima from central one.	.014	.013	.014	cm.
On the other side	.013	.014	.015	cm.

Calculated value = .015 cm.

The central fringe was always found to be the darkest of the system as pointed out in the theory.

For the verification of the second case, such a stria was selected in the mica, which showed two steps close together, separated by a bright region. The same high power objective was used as in the previous experiment, and the angle 2θ , was determined by measuring the distance between the edges in the focal plane of the microscope by the micrometer eyepiece, and dividing it by D . The following data were obtained: $\frac{2\pi}{\lambda} \theta \xi = 630$ nearly

$\frac{2\pi}{\lambda} \theta_1 \xi = 4.08$, $D = 30.0$ cms., $2\xi = 201$ cm. The method of illumination was the same as in the previous case.

We find from Glaisher's tables that the difference between the values of $\text{si}(630)$ and $\text{si}(600)$ is small enough to be negligible for our purpose, so for simplicity $\theta_1 \xi$ was taken equal to 600, and $\theta_1 \xi = 4$ approximately. Fig. 7 shows the positions of maxima and minima for this case. The following table shows the agreement between the observed and calculated values:—

$\lambda = 5 \times 10^{-5}$ cm. $\theta_1 \xi = 4$, $\theta_1 \xi = 600$ approximately.

Serial No.	1	2	3
Observed distance between successive minima (on one side).	·0145	·0145	·0160 (very faint).
On the other side	·0145	·0150	

Calculated value = ·0146 cm.

The distance between the geometrical images in the focal plane was ·0195 cm. and $D = 30.0$ cm.

It is to be noted that it was difficult to judge the point of critical focus, and very slight changes in the position of the focus, produced appreciable errors in the measurements taken. The fringes were extremely fine, and the finite width of the moving wire of the micrometer also produced a slight error in the measurement. Considering these facts, the observed and calculated values show a close agreement.

In both these cases, the diffraction pattern showed an asymmetrical character, the fringes on one side being clearer and more in number than on the other side. This is not accounted for by the theory developed above, but nevertheless the theory explains the observed facts to a close degree of approximation.

For the third case, the boundaries studied were that of mixed plates formed of albumen and air bubbles.¹ When these mixed plates are allowed to dry, the albumen forms narrow ridges, separ-

¹ C. V. Raman and B. N. Banerjee, loc. cit.

ated by broad air spaces on either side examined under the microscope, the ridges exhibited effects, which showed good qualitative agreement with the theoretical results indicated in Figs. 10 and 11. No quantitative measurements were, however, taken in this case as the exact configuration of the ridges is to some extent still uncertain.

VII.—EFFECTS OF OBLIQUE ILLUMINATION.

When a single stria is focussed under the microscope, its axis being horizontal, and the light being incident normally on it, the fringes on either side bordering the sharp black image of the stria are equidistant, the only difference being noticeable is in their visibility, they, on one side, in a particular case, on the left hand portion of the field of view being more clear than on the other. As the whole microscope together with the stria is turned slowly towards the left, i.e. in the direction in which the fringes apparently appear sharp in the field of view, and its focus is kept unaltered, so that light is now incident obliquely on the stria, it is seen that the distance between the consecutive fringes on the left increases, and they become broader and diffused, whereas the fringes on the right side appear to come closer together, and become more sharp and fine. This phenomenon persists till the direction of the incident beam is very oblique.

When the whole microscope is turned gradually in the opposite direction, i.e. towards the right, the fringes on the right hand portion of the field of view of the observer, become more distant and diffused, whereas on the other side, they come closer together, and become very fine and sharp. The same changes are also noticeable in the case of a more complicated stria, consisting of more than one step, and the mixed plates mentioned above also show increased asymmetry with oblique illumination than with normal illumination.

VIII.—CONCLUDING REMARKS.

In his theory of the phenomena observed in Foucault's Test, Rayleigh¹ has investigated the distribution of intensity in the focal plane of an observing telescope, under various conditions of limit-

¹ Rayleigh, *Phil. Mag.*, Feb., 1917.

ations of aperture of the objective, when a transparent object introducing a phase-retardation is viewed by the well-known 'knife-edge' test. The conditions involved in the theory of the method are not dissimilar to those obtaining in the theory of microscopic vision, and detailed analysis practically leads to the same results as the theory developed in the present paper. Both these theories, however, fail to explain the asymmetrical character of the effects actually exhibited in practice by laminar boundaries. The assumption that the effect of a thin laminar boundary is merely to impose a simple path-retardation on a part of the wave front is probably not quite accurate. The exact configuration of the laminar edge is undoubtedly of great importance in determining the character of the light diffracted by it through large angles. The influence of the shape of the laminar edge is particularly well-exhibited in the case of 'mixed plates,' where the edges are drawn inwards under the action of surface-tension, and a considerable amount of light is scattered at these edges.

In this connection, mention may also be made of some observations made by the writer on the effects of a certain type of fungus growth on optical glass, the edges of which exhibited the phenomena of laminar diffraction exceedingly well. The fungus shows gorgeous colours, when examined by light scattered by it in different directions, the colour depending on the angle of diffraction. When a particular edge of the fungus was singled out for observation, it showed marked asymmetry in its colours as observed from either side. The influence of this asymmetrical diffraction on the microscopic appearance of the edges remains yet to be mathematically investigated.

Another case of great importance in the theory of microscopic vision is that of laminary gratings; so far as the writer is aware, no attempt has been made to apply the rigorous theory of gratings as developed recently by Rayleigh¹ and Voigt² to the interpretation of the microscopic appearance of grating structures. The writer hopes at an early opportunity to take up this problem.

The work was carried out in the laboratory of the Indian Association for the Cultivation of Science, and the writer wishes to

¹ Rayleigh, Scientific Papers, Vol. V, pp. 388-404

² Voigt, Gottingen Nachrichten, 1910. It may be mentioned that Wolfke has dealt with the case of a *simple* laminary grating.

record his sense of deep gratitude to Prof. C. V. Raman for his encouragement and guidance during the progress of the work.

210, Bowbazar Street,
Calcutta, India,
5th May, 1921.

XIV. On the Theory and some Applications of Sub-synchronous Pendulums.

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CONTENTS.

- I.—Introduction.
- II.—Chronographic Determination of Frequency.
- III.—General Characters of Sub-synchronous Maintenance.
- IV.—A simple Theory of Sub-synchronous Maintenance.
- V.—Discussion of the Mathematical Theory.
- VI.—Summary and Conclusion.

I.—INTRODUCTION.

When an iron rod swings as a pendulum over a vertical bar-electromagnet through which is caused to pass an electric current rendered intermittent by an interrupter, the pendulum is sometimes found to be maintained in a state of permanent and vigorous oscillation against the natural dissipative forces tending to bring it to rest. The magnetic force does not tend to displace the pendulum appreciably from its equilibrium position when the pendulum is at rest and so the pendulum requires an independent starting, generally with a large amplitude, from which it settles down to the amplitude requisite for maintenance.

The phenomenon was noticed and investigated mathematically by the late Lord Rayleigh¹ for the case of a pendulum whose frequency was $\frac{1}{2}$ that of the interrupter. Recent work by Mr. Dey and Prof. C. V. Raman² has extended the limit of the frequency ratio to a wide limit, inasmuch as, the maintained oscillations of the pendulum may have a frequency which may be any one of the series of fractions $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}$, etc., up to $\frac{1}{10}$ of the frequency of the interrupter fork and the series is capable of being

¹ Scientific Papers. Also Theory of Sound, Vol. I, p. 82.

² Proc. Roy. Soc. A., Vol. 95, 1919.

extended to still smaller ratios for the possible frequency of maintenance. These authors have further brought out certain interesting features in the working of the sub-synchronous pendulum, namely, the variation of the amplitude of the pendulum over the possible range of frequency of maintenance, the sequence of which is totally unlike the ordinary type of resonance curve of a simple vibrator under the action of a periodic force. The paper quoted discusses the application of the pendulum to the absolute determination of acoustical frequencies and an Appendix to it contains the outline of the dynamical theory of maintenance. It is proposed in the present paper (*a*) to test the exactness of the numerical ratios of the frequency of the pendulum and fork-interrupter and to make by chronographic methods an accurate determination of the frequency of the fork interrupter; (*b*) to give a simple explanation of the fact that the maintenance occurs in even submultiple ratios of frequencies and of the other features of maintenance observed by the present writer and (*c*) to test by actual measurements of energy losses and gains of the pendulum on the theory of sub-synchronous maintenance given by Prof. C. V. Raman.

The experiments described in the following pages have been done with a pendulum 35 cms. long with a heavy brass bob attached to it; the suspension for the pendulum being a loop of wire working in two wire hooks. The electromagnet placed just below the pendulum rod had an iron core 12 cms. long and was fed by interrupted currents (1.2 amp.) from a fork of frequency $\cdot 32 \sim$. The fork was maintained in successful oscillation for hours together by an electromagnetic arrangement, the sparking of it being smoothed by shunting a microfarad condenser across the spark gap, and submerging the mercury surface of the interrupter in a layer of absolute alcohol.

II.—CHRONOGRAPHIC DETERMINATION OF FREQUENCY.

It has been noticed in the introduction that for the maintained oscillations of the pendulum its frequency may be any one of the series of fractions $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, etc., of the frequencies of the fork interrupter. The exact equality of the ratio of the frequency of maintenance of the pendulum to that of the interrupter fork to one or other of the fractions mentioned may be demonstrated by

various methods, and it may be shown that the pendulum follows accurately even the slight variations in the frequency of the fork, caused by fluctuations in the room temperature. This has been done by Mr. Dey in the paper quoted by three methods (i) by comparison with standard pendulum clock by visual observation of coincidences; (ii) by comparison with the ticks of a half seconds standard chronometer; (iii) by allowing the sub-synchronous pendulum to run the hands in a clock dial of which the rate can be determined against a standard time keeper.

The present writer has adopted another method which is slightly more elaborate but is nevertheless easy to work and yields accurate results. The frequency of the fork is determined accurately by finding the exact time in which the sub-synchronous pendulum executes, say, one thousand oscillations; this is done with the aids of the time signals given by the pendulum on a chronograph tape at the beginning and at the end of an interval of few minutes. The seconds-pendulum of the Electric-Master-Clock in the Laboratory furnished an excellent standard being previously rated to keep time accurate to a second a day. The hammer attached to its mechanism, by half-minute impulses kept the pendulum in oscillation and the time was indicated by corresponding half-minute ticks. In order to get seconds-signals on the chronograph, an electric circuit through the pendulum rod was arranged which was completed in each swing of the pendulum by a platinum wire pinned to the bottom of the pendulum rod making contact on a mercury cup placed at the centre of swing. A similar electric circuit was arranged for records of the half periods of the sub-synchronous pendulum. The chronograph recorder supplied by the Cambridge & Paul Instrument Co., Ltd., had ink styles worked by relay circuits and the moving tape could be started and stopped by the action of a lever on the revolving gear.

The experimental procedure is simply to get a few records of the pendulum and the standard clock side by side on the chronograph tape just when the clock hammer once drops, then wait for the next 22nd drop, when a few records are again taken. Thus there is a clear interval of 11 minutes between the successive records at the beginning and at the end. The chronograph gear is usually started earlier than the appointed time, so that the inter-

val between dropping of the hammer and working of the lever is practically negligible. The temperature of the fork under observation is recorded by a sensitive thermometer placed midway between the prongs of the fork. The values thus obtained of the temperature, though they might differ slightly from the actual temperature of the fork itself, have been taken to be correct, as not much deviation from the mean value was observed during a single experiment.

TABLE I.

Serial number of observations.	Frequency-ratio.	Observed correction from record (Sec.).	Temperature of fork (degrees cent.).	Calculated frequency of fork.	Frequency of fork reduced to 26°C.
1	$1/80$	-0.063	25.20	32.00152	31.99995
2	$1/80$	-0.211	25.46	32.00113	31.99987
3	$1/36$	-0.246	25.92	31.99998	31.99996
4	$1/82$	+0.002	26.00	31.99990	31.99990
5	$1/82$	-0.054	26.20	31.99956	31.99991

Table I shows a few results of determination of the frequency of an aluminium fork (of frequency 32 ~) while maintaining in sub-synchronous oscillation a pendulum (of length 35 cms.). It will be noticed that working with three different frequency ratios and five independent positions of the bob, it is possible to get the frequency of the fork correct to a few parts in a million.

The method of calculation was as follows—The loss or gain of the clock for an integral number of oscillations of pendulum was measured from the cross-marks on the records at the beginning and at the end. Thus the time for a complete number of oscillations of the pendulum was known. To obtain the whole number of oscillations of the fork during the interval, the loss or gain in sec. (the correction term in Table I) was divided by the rough period of the fork (0.031 sec.) and the integral part so obtained was added to or subtracted from 21, 120 (the number of oscillations of the fork in

11 min.) according as the pendulum gained on the clock or the clock gained on the pendulum.¹

III.—GENERAL CHARACTERS OF SUB-SYNCHRONOUS MAINTENANCE.

With the experimental arrangements adopted, maintenance is obtained with amplitudes so large that the ordinary theory of small oscillations is not applicable. Moreover, the remarkable result is that while maintenance has been obtained at a large amplitude and the pendulum is released (e.g., by a sudden push to one side), it is not 'caught' at the amplitude at which it was maintained originally and the pendulum after executing a number of damped oscillations, is again caught at a smaller amplitude. E.g. it has been noticed that while the maintained amplitude was 49° and the pendulum made 150 oscillations in 85.1 sec. being released, it was again maintained at 35° with 150 oscillations in 82.4 sec. and then again at 10° with 150 oscillations in 80.4 sec. This point has been tested carefully with various other positions of the bob leading to the general result that maintenance is possible with two or three amplitudes when the natural frequency of the pendulum and that of the interrupter are not interfered with.

Maintenance of a pendulum in sub-synchronous oscillation is generally obtained with amplitudes ranging between 10° and 60° , with the experimental arrangements adopted; as for amplitudes less than 10° , the pendulum departs much from the plane of its motion, being often irregular apparently due to the proximity of the metal rod in vibration in the magnetic field and the consequent unavoidable generation of eddy currents in its mass. The manner in which the amplitude of the forced oscillation of the pendulum varies by varying the position of the bob in the rod so as to alter its natural period of oscillation shows the peculiarity of increasing continually to a large value as the position of the bob in the rod is raised till a stage is reached at which the pendulum

¹ Another procedure is, of course, to divide the corrected time by the period of the pendulum, determined sufficiently accurately by means of a stop watch, and multiply the integral number so obtained by the denominator in the frequency ratio, determined from a knowledge of the periods.

refuses to be maintained. This is followed by another region of maintenance, and so on.

As illustrative of the nature of variation, some results obtained by Mr. Dey are quoted below.¹ The writer will have occasion to refer to them in a later section, as leading to a qualitative support of his theory.

TABLE II.

Frequency-ratio								
$\frac{1}{20}$	{	<u>free period of the pendulum</u>	..	19.4	19.6	19.8	20	20.1
		period of the fork	..	40°	32°	24°	18°	14°
$\frac{1}{22}$	{	<u>free period of the pendulum</u>	..	21.5	21.8	22		22.4
		period of the fork	..	44°	32°	24°		12°
$\frac{1}{24}$	{	<u>free period of the pendulum</u>	..	23.5	23.8	24		24.6
		period of the fork	..	46°	32°	24°		8°
$\frac{1}{26}$	{	<u>free period of the pendulum</u>	..	25.8	26	26.5		27
		period of the fork	..	40°	32°	16°		8°

Still another result of interest in the maintenance to be noted is the relation between the period of the pendulum while vibrating freely in the field of gravity and the periods of maintenance with different amplitudes. The experimental results obtained point to the fact that the free period of the pendulum is decreased by the presence of the electromagnet.

¹ The figures in Table II are roughly gathered from fig. 1, p. 536, Proc. Roy. Soc. A Vol. 95, 1919.

Fig. 1 shews the free and forced periods of oscillation plotted against successive amplitudes. The free period of the pendulum at a definite amplitude was determined by taking the mean value of the time for a number of oscillations while the pendulum was executing damped oscillations about the amplitude in question.

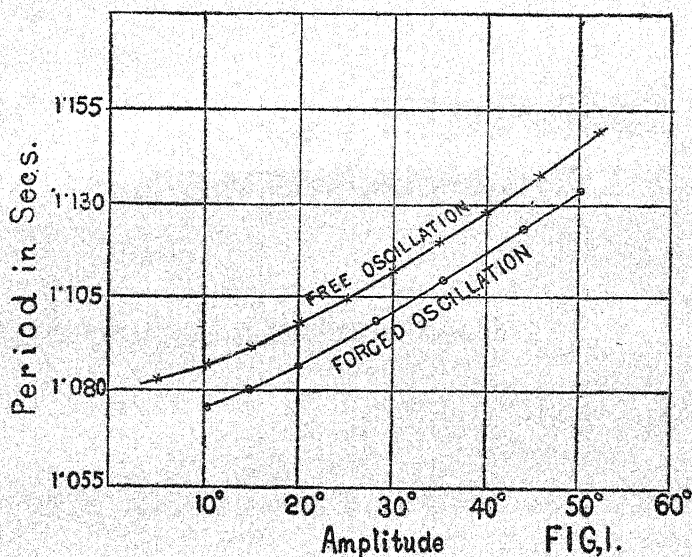


FIG. 1.

The forced periods represent the periods of maintenance obtained at different amplitudes by loading the fork with small weights so as to alter its frequency. It will be noted that the free and forced periods do not differ much, a result of much theoretical significance, as will appear from the discussion of theory given later on.

Lastly, it may be remarked that the difficulty sometimes experienced in getting maintained oscillations of the sub-synchronous pendulum, is attributable to the want of exact phase relationship between the oscillations of the pendulum and that of the magnetic field. Experience shews that maintenance is most likely when the pendulum receives the least jerking when passing over the electromagnet; thus shewing that it passes its equilibrium position a little before or after the maximum phase of the field.

The actual phase difference has been studied for a number of cases, by recording on a moving photographic plate the oscilla-

tions of a steel blade by reflecting a spot of light from a small mirror fixed on to it. The blade is fixed horizontally in the neighbourhood of the electromagnet by a bifilar suspension under tension, adjusted for resonance. The pendulum while passing through its equilibrium position occults the moving spot of light. The actual phase difference can then be measured from the occultation phase,—knowing that the oscillations of the spot of light follow those of the field by quarter of an oscillation. Thus it has been found that the difference of the phases of the pendulum and the field which is almost zero for small amplitudes becomes greater and greater till it attains the value $\frac{\pi}{2}$ for very big amplitudes of maintenance.

IV.—A SIMPLE THEORY OF SUB-SYNCHRONOUS MAINTENANCE.

That the behaviour of the pendulum in sub-synchronous oscillation is unlike that of a vibrating system which when acted on by a periodic force, ultimately vibrates in a period the same as that of the force, is apparent; but the analogy becomes clear and close on consideration of the fact the difference in the present case is due to the adaptability of the pendulum to the low period

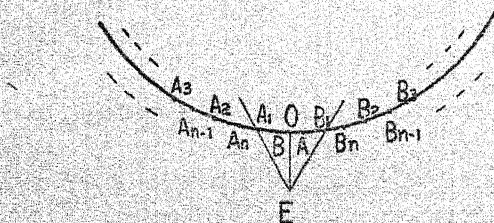


FIG. 2.

FIG. 2.

of the fork interrupter limited by the range of adjustment of the period of the pendulum to the case of exact synchronism; and thus the pendulum vibrates so as to be synchronous with an even multiple of the period of the fork to which it happens to be near by reason of its own limited free period. Why the multiple is always even will appear from considerations given below.

As a characteristic of this form of forced oscillation, it will be sufficient to remark that previous to the stage of steady main-

tenance, the vibration is intermittent with beats whose frequencies equal the difference of the frequencies of the system and the force.

As the frequency of the fork interrupter and hence of the periodic field is fairly high in comparison with that of the pendulum, we might consider the impulses received by the end of the pendulum rod as sudden enough to be represented to have taken place while the pendulum is in isolated positions on its arc of swing XBAY (fig. 2) separated by finite intervals, equal to the period of interruptions, such as $A, A_1, A_2, \dots, A_{n-2}, A_{n-1}, A_n; B, B_1, B_2, \dots, B_{n-2}, B_{n-1}, B_n$. XEY represents the limited region within which the attraction of the Electromagnet is perceptible,—usually small in comparison with the big amplitude of maintenance obtained in practice.

The conditions requisite for maintenance, in view of the fact that a balance of impulse must be carried forward by the pendulum on both ways of its motion, through the zone of electromagnetic attraction, may thus be found out. It might reasonably be assumed that the attractions by the electromagnet of the pendulum rod are symmetrical¹ about EO, and fall off gradually in intensity till they vanish at X and Y. Further if for the positions A and A_1 , B and B_1 , $OA_1 > OA$ and $OB_1 > OB$, the cumulative effect of impulsive attractions of gradually increasing intensity received in positions B_{n-1}, B_n and A outweighs the counter attractions received in positions A_1, A_2, A_3 , and a balance of impulse is carried forward. Similarly for motion of the pendulum towards Y, an identical state of things happen if $BO = OA$. In actual practice, the number of positions included within the zone of electromagnetic attractions are few (not more than four in all) and so the case is much simpler than that discussed here.

The number of A_1, A_2, \dots, A_n and B_1, B_2, \dots, B_n positions are evidently equal from symmetry and is each an integer (1, 2, 3, etc.) depending on the actual angular amplitude at which maintenance is obtained. Thus taking the two positions A and B

¹ That the impulses received by the maintained pendulum are of the nature contemplated, may be gathered by suspending the pendulum from the side of a projecting wooden beam, when the rhythmic character of the reactions on the support due to symmetrical impulses, would be plainly heard on putting the ear close to the projecting end of the beam, to afford a striking contrast to the irregular jerking at stages when maintenance is not obtained.

into consideration, the total number of such positions in a period of the pendulum is $2n+2$; which is always even. Thus maintenance is obtained only for even submultiple ratios of the periods of the field and of the pendulum and hence of their frequencies.

Frequency ratio and amplitude.

From the conditions already obtained for maintenance, it follows that the frequency of the interruptions being kept constant, maintenance might be obtained of the same pendulum at various amplitudes so as to alter the frequency ratio by successive additions or subtractions in its denominator of the number 2, as we can make additions or delete equal portions of the arc $XBAY$ (fig. 2) from its ends so as to include or exclude every time a single position of the pendulum on each side, without interfering with the maintenance. These deductions of theory fully explain the facts of observation already noticed that maintenance might be obtained of the same pendulum for two or more amplitudes, the actual drawback in obtaining the theoretically great number of such maintenances lies in the fact that the phase of passage of the pendulum in relation to the periodic magnetic field differs for different angular amplitudes of maintenance, thus necessitating a readjustment of the positions of affairs within the zone of the electromagnet, and making actual adjustment for maintenance a matter of chance.

Free period of the pendulum and amplitude.

The experimental results obtained as to the variation of amplitude of maintenance for the same frequency ratio admits of a simple explanation in the light of the theory indicated above. The general results obtained both by the author and Mr. Dey, shew the increase of amplitude for decrease in the free period. (*Vide* Table II) The period of the interrupter remaining the same, it would follow from fig. 2, that if the period of the pendulum be increased the actual distances between consecutive positions A_1 , A_2 , A_3 , etc., must decrease, to keep the number of such positions to a defined integral value for the same frequency ratio; thus decreasing the amplitude for maintenance.

Phase-difference between the field and the pendulum.

In the diagrammatic scheme (fig. 2) we have assumed that O, the position of equilibrium of the pendulum lies assymmetrically

with regards to positions A and A_1 or B and B_1 ; or in other words the pendulum passes its equilibrium position, assymmetrically with regard to two successive positions of maximum magnetism. Photographic studies of phase confirm these hypotheses, as assymetry is considerable for comparatively large amplitudes of maintenance. As regards small amplitudes of maintenance the assymetry is less marked and the pendulum passes its equilibrium position at about the midway between two maxima. These facts explain to a great measure, the slight attractive forces of the electromagnet for smaller amplitudes and greater attractive for bigger amplitudes, thus making the decrease in the free period nearly the same for bigger than as for smaller amplitudes as in fig. 1.

V.—DISCUSSION OF THE MATHEMATICAL THEORY.

It has tacitly been assumed in developing the theory already given that the energy lost by dissipation is just compensated by the flow of energy from the field. The dynamical theory given by Prof. C. V. Raman admits of quantitative evaluation of the flow and the loss by dissipation. The system having one degree of freedom its equation of motion is written in the form

$$\ddot{\theta} + k\dot{\theta} + [n^2 - \sigma\theta^2 + f(t)F(\theta)] \theta = 0 \dots\dots\dots (1)$$

Now assuming the maintained oscillation of the pendulum to be represented by

$$\theta = \psi_1 \sin (\rho t + \epsilon_1) \dots\dots\dots (2)$$

it can be easily shown that the rate of loss of energy due to dissipative forces

$$= \frac{1}{2} k \rho^2 \psi_1^2 t \dots\dots\dots (3)$$

and the energy which flows from the field to the pendulum, on the assumed mode of vibration in which $m=2s\rho$, where S is an integer,

$$= \frac{1}{2} \rho \psi_1^2 \gamma F(\theta) \left[\frac{\sin (2\rho + m)T \cos \frac{m}{\rho} \epsilon_1}{2\rho + m} - \frac{\cos (2\rho - m)T \sin \frac{m}{\rho} \epsilon_1}{2\rho - m} \right]$$

where

$$T = \frac{1}{\rho} \sin^{-1} \frac{\rho}{\psi_1} \dots\dots\dots (4)$$

ϕ being the angular distance within which the force of the electro-magnet is perceptible. The constants in the expressions (3) and (4) may be determined in the following way.

K.—The logarithmic decrement of oscillations of the pendulum is determined, while it is being damped by the interrupted field of the magnet about the amplitude at which maintenance is obtained. *K* is known from the relation $\log\text{-decrement} = \frac{1}{4} KT$, where *T* is the period of the pendulum for the amplitude of maintenance. (The logarithmic decrement instead of being found by the tedious process of recording successive amplitudes, is determined quickly without seriously affecting the accuracy from the time (*t*) in which an amplitude falls to $\frac{1}{e}$ th its value, the log-decrement being then equal to $\frac{2T}{t}$).

ϕ and ψ_1 are given by $\frac{2\pi}{\text{period}}$ and the circular measure of the amplitude respectively.

$\gamma F(\theta)$ —to one end of a spring (a hack-saw blade) a few cms. of the pendulum rod is fixed by soldering (the frequency of the combination being about 60 per sec.) This is clamped in a vice and hangs over the electromagnet, with the rod in position of the pendulum. The interrupter being worked the amplitude of oscillation of the spring is measured on the scale of a microscope focussed onto a fine luminous point on the rod. The deflecting force corresponding to the amplitude is measured by deflecting it statically by known weights placed on a scale pan attached to the pendulum rod by a string passing over a pulley. The actual value of the deflecting force or in other words, the amplitude of the oscillating spring varies slightly with different angular positions of the spring with respect to that of the electromagnet, and the mean value of several such deflecting forces is taken as the measure for $\gamma F(\theta)$.

ϵ_1 —Determinations are made as described in Section 3, page 149

ϕ and $m - \frac{\phi}{m}$ being equal to the 'frequency ratio' of maintenance is known by determinations of the period of the pendulum that of the interrupter being known from previous determinations.

T is calculated from the relation $T = \frac{1}{\phi} \sin^{-1} \frac{\phi}{\psi_1}$ where ϕ is determined approximately by deflecting the pendulum by weights attached to the end of a string passing over a pulley, while (i) the interrupter is working, (ii) the interrupter is not working. The meeting point of the two curves to represent the deflecting weight against deflection, on the same scale gives the value of ϕ . Thus it has been calculated that for the case $m=36 \phi$.

Amplitude of maintenance.	Energy lost by dissipation in ergs per sec.	Energy gained from field in ergs per sec.
39°	0.231	0.264
22°	0.071	0.082
12°	0.016	0.020

VI.—SUMMARY AND CONCLUSION.

1. The maintained oscillations of a subsynchronous pendulum which was noticed and investigated by Lord Rayleigh and later by Mr. Dey and Prof. C. V. Raman, have been studied with a 35 cms. iron-rod pendulum and an interrupter fork of frequency 32.

2. The exact equality of the ratio of the frequency of maintenance of the pendulum to that of the interrupter fork to one or other of the 'frequency ratios' for slight variations in the temperature of the room, is established by measurement of the absolute frequency of the fork by chronographic records.

3. Some of the characteristics of subsynchronous maintenance are—

(a) maintenance of subsynchronous pendulum may be obtained at two or three different amplitudes, when the position of the bob on the pendulum and the frequency of the interrupter are not interfered with.

(b) maintenance is not possible for very large and for very small amplitudes.

(c) the free and the forced period of the subsynchronous pendulum for different amplitudes, obtained by altering the frequency of the interrupter differs but slightly from each other, the difference being the same for big as for comparatively small amplitudes.

(d) Results obtained by altering the position of the bob on the other hand, shew increase of amplitude of maintenance for decrease in the free period.

(e) A method has been developed for determining by photographic record, the phase of passage of the pendulum through its equilibrium position in relation to the periodic magnetic field. The phase-angle is almost zero for small amplitudes and attains the value $\frac{\pi}{2}$ for very big amplitudes.

4. From a simple theory of subsynchronous maintenance on the basis of the impulsive nature of the attraction of the electromagnet on the pendulum and the limited zone of action of the electromagnet through which energy required to counteract dissipation is supplied, the condition requisite for maintenance is found to be that the two consecutive attractions near the equilibrium position of the pendulum are asymmetric about the mean-position. The theory explains primarily the even submultiple ratio of frequencies necessary for maintenance. It explains the facts of observation in 3(a) and 3(d). The assumed asymmetry of theory is brought out by the photographic records of phase and the nature of asymmetry observed explains the result noticed under 3(a).

5. The expressions of the mathematical theory for the dissipation and the flow of energy from field, have been evaluated for a few cases and there is a fair agreement between the two values.

The work was carried on in the Laboratory of the University College of Science and the writer, in conclusion, wishes to express his best thanks to Prof. C. V. Raman for suggestions and guidance.

XV. On Whispering Galleries.

By C. V. Raman, M.A., Hon. D.Sc., Palit Professor of
Physics in the Calcutta University.

(Plates X, XI, XII, XIII and XIV.)

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- II.—Description of the Gol Gumbaz at Bijapur.
- III.—The Victoria Memorial at Calcutta.
- IV.—The Granary at Bankipore (Patna).
- V.—Whispering Gallery at the Calcutta G.P.O.
- VI.—Propagation of Sound in Whispering Galleries.
- VII.—Synopsis.

I.—INTRODUCTION.

In the volume of collected papers on Acoustics by the late Prof. W. C. Sabine published recently by the Harvard University, there is a very interesting article on 'Whispering Galleries' in which the architectural and acoustical features of several remarkable structures in Europe and America have been discussed. No mention is however made of whispering galleries in other parts of the world. Among the Indian whispering galleries the most notable, architecturally and acoustically, is the great Gol Gumbaz at Bijapur. In the Victoria Memorial recently completed at Calcutta, there are two very fine whispering galleries, one of which, curiously enough, remained unsuspected till it was discovered by the writer. There is also another whispering gallery at the Calcutta G.P.O., also first noticed and studied by the writer. The acoustical properties of the building known as the Government Granary at Bankipore in Patna District are also of much interest. The present paper contains a description of these whispering galleries. Other acoustical curiosities, such as Sekundar's tomb at Fatehpur Sikri near Agra, have been brought to the writer's notice, but they are not here discussed.

While in England in the year 1921, the writer in collaboration with Mr. G. A. Sutherland of the University College, London, had an opportunity of carrying out a study of the well-known whispering gallery in the Dome of St. Paul's Cathedral at London, and the results of the investigation showed the presence, with a steady source of sound, of an interference-field in the gallery with radial and circumferential nodal lines.¹ The comparative study of the three whispering galleries at Calcutta conveniently accessible to the writer has brought to light some further facts of interest relating to these structures. These are also discussed in the paper.

II.—THE GOL GUMBAZ AT BIJAPUR. (See Plate X.)

“Transcending all other buildings at Bijapur in simple mass, and dominating the landscape for miles around, the great Gol Gumbaz or tomb of Sultan Muhammad,² stands alone (Plate X). For size, few other buildings in India can be compared with it. Its noble proportions and magnificent dome are only seen to the fullest advantage from a distance. When close up to it, the dome seems to sink into the building, and to require an intermediate terrace or storey to raise it into full view. A few extra feet here would certainly have improved the general design, even when viewed from further off. The impressive grandeur of the building and its imponderable mass simply overwhelm the spectator with awe. It stands in the extreme east end of the city, its massive basement resting upon the solid rock. The vast mausoleum stands out with most striking effect when viewed, as Muhammad himself must often have seen it, from the upper hall of the Athar Mahall, when, backed by great storm clouds, the low western sun suddenly bursts through a rent and illumines the great building. It then flashes out into brilliant contrast against the rolling masses of angry black clouds; the mellow tints of its walls are bathed in a golden glow, and the great dome shines like burnished brass. Under all this glory peacefully repose the remains of Sultan Muhammad.

“King Ibrahim, his father, had raised the beautiful pile of the Ibrahim Rauza, which was the last word in decorative and luxurious magnificence. It was impossible for Muhammad to go further

¹ Nature, Sept. 1921, and Proc. Roy. Soc. 1921.

² King of Bijapur in Southern India, seventh of the Adil Shahi Dynasty, from 1627 to 1656.

upon the same lines, so he struck out in a different direction altogether, and endeavoured to dwarf it, and everything else, by stupendous mass; and this he certainly accomplished. The Gol Gumbaz is the antithesis of the Ibrahim Rauza in that the strong virility of conception of the one contrasts with the delicate femininity of the other. His reign of thirty years, however, was not sufficient wherein to fully complete the building, for he seems to have died while the plastering of the walls was in progress, and it was no one else's business to complete what he left unfinished. One cannot help wondering what new departures would have

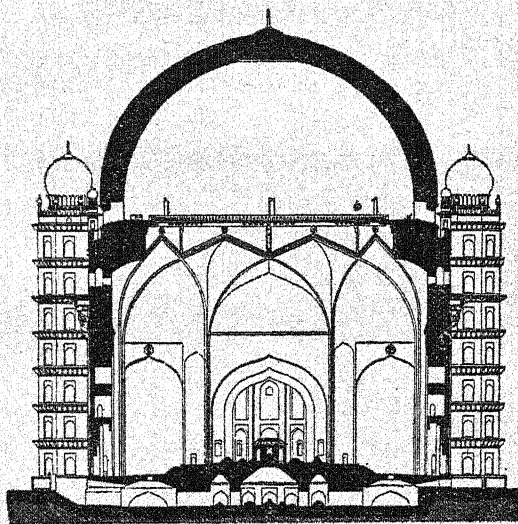


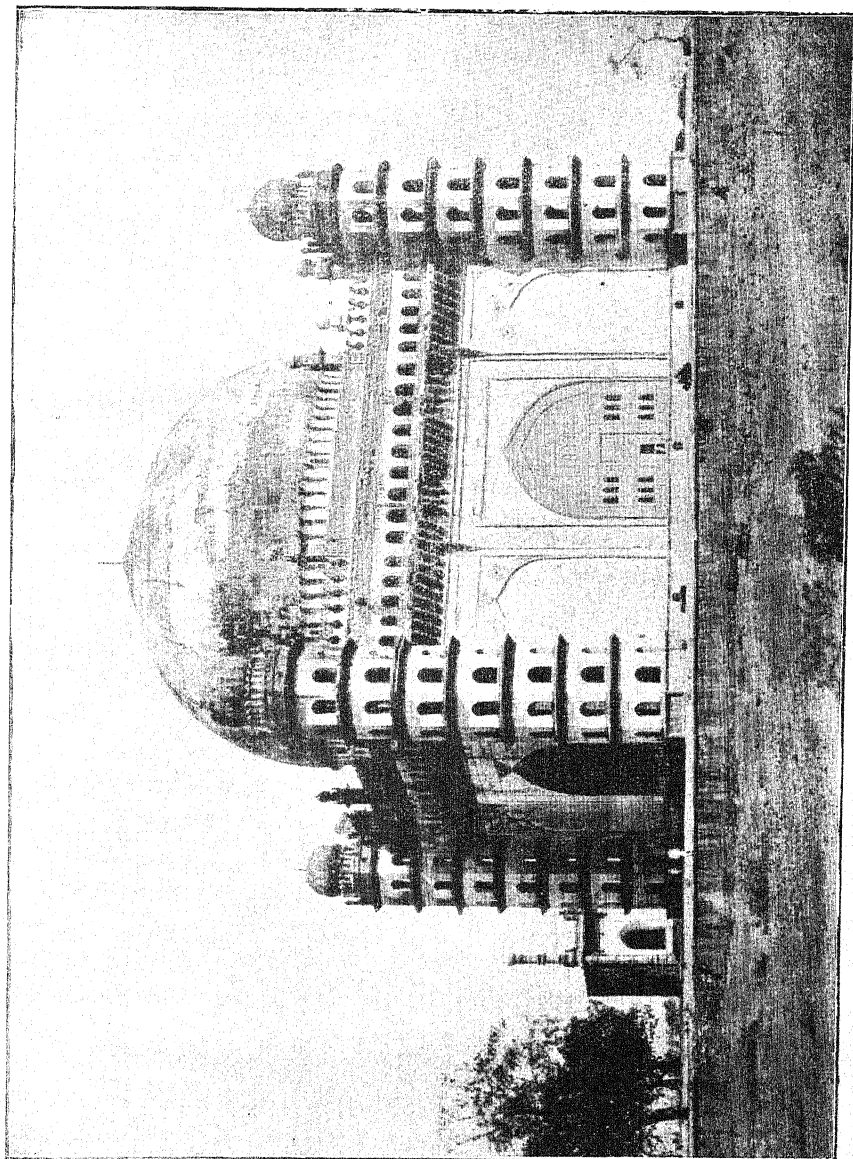
FIG. 1. The Gol Gumbaz at Bijapur (Architectural Drawing).

been made in the further development of Bijapur architecture had the dynasty lived and flourished another hundred and fifty years, for they were daring builders.

“A glance at the plan (Fig. 1) of the Gol Gumbaz shows what a simple building it is for all its size—just a great square hall, enclosed by four lofty walls, buttressed up by octagonal towers at the corners, and the whole surmounted by a hemispherical dome. The great size of the dome, and the neat and perfect manner in which, by means of cross arching and pendentives, the square walls have been worked up to meet it, are the most notable features of the building. The extreme outside measurement of the mauso-

leum including the towers is 205 feet square. The extreme height to the apex of the dome from the base of the building is 198 feet 6 inches; the exterior diameter of the dome is 144 feet while the interior diameter, measured 124 feet 5 inches; and the great hall, below, with no intermediate supports of any kind, inside its walls, is 135 feet 5 inches square. The interior height, from the level of the floor, around the tomb platform to the top of the dome is 178 feet. Within the base of the dome is a broad gallery, 11 feet wide, which hangs out into the interior of the building, 109 feet 6 inches above the floor. Narrow staircases wind up through the corners of the building where the towers join it, and passages lead out from them on to each of the pigeon-holed storeys of the towers. In the centre of the floor of the hall is a high platform upon which are the counterfeit domes, the real graves being in the vault underneath which extends over the whole length and breadth of the hall.

"Each of the four walls of the building had been raised as three great arches, the central one being wider than the two side ones, and these have been filled in with rubble masonry in the side ones, and cut-stone in the central one. On the north side, however, the central archway had been left open, or had been subsequently opened, as a small chapel or chamber has been built against the wall here as an annexe, communicating through the arch with the great hall within. In the central archways of the other three sides are the doors and windows; but, even here, the filling in above the windows can only be called crazy or patchwork-masonry. The masonry of the great dome may be looked upon as practically concrete it being composed of bricks in mortar, the thickness of the shell varying from ten feet at the springing to nine feet near the crown. It is thus a great rigid concrete shell without voussoirs, and, consequently, with practically no side thrusts of any kind so long as it remains intact. It is a dead weight acting vertically downwards partly upon the cross arching within and partly upon the side walls just as an inverted china basin would act upon the upper edges of a cube upon which it might rest. Being built in this way, with ring upon ring of thick brick work, each corbelled forward until they close at the apex, it is probable that no centering or support was needed beneath it during construction, except, perhaps, for a small section near the crown, which would have been supported by that part of the dome



The Gol Gumbaz at Bijapur.

already built. An outward thrust that could possibly come upon the side walls would be amply neutralised by the weight of the material in the pendentives which hang over inside the building.

"This system of pendentives is, without doubt, the most successful and most graceful method of construction for such domes. It obviates any interference with the external contour of the dome, and adds, at the same time, a very pleasing feature to the interior—the interlacing or groining of the arches. The tendency of the dome to spread at the base, which is counteracted by the pendentives and great mass of masonry thus thrown into the interior of the building, was guarded against, in case of the Pantheon at Rome, which possesses a dome of greater diameter, by the heaping up of masonry upon the haunches of the dome outside, thus destroying its beauty of outline.

"The great hall below, which is covered by the dome, covers an area of 18,337·67 square feet, from which if we take 228·32 square feet for the projecting angles of the piers carrying the cross arches, which stand out from the walls into the floor, two on each face, we get a total covered area, uninterrupted by supports of any kind, of 18,109·35 square feet. This is the largest space covered by a single dome in the world, the next largest being that of the Pantheon at Rome, of 15,833 square feet. If we add the pendentives to the actual dome, to which they naturally belong as part of the superstructure, this then becomes the greatest domical roof in the world.

"But, was not this great dome, after all, but an afterthought?.....Before the walls of the Gol Gumbaz had risen many feet, it would seem that the plans were altered. The daring spirit of the architect, urged on perhaps by Sultan Muhammad himself, incited him to attempt the more stupendous task of hanging a mighty dome right across the whole expanse of the outer walls; and it seems almost incredible that the man who conceived, and carried to such a successful issue, this magnificent project, should have passed into oblivion; his very name is unknown.

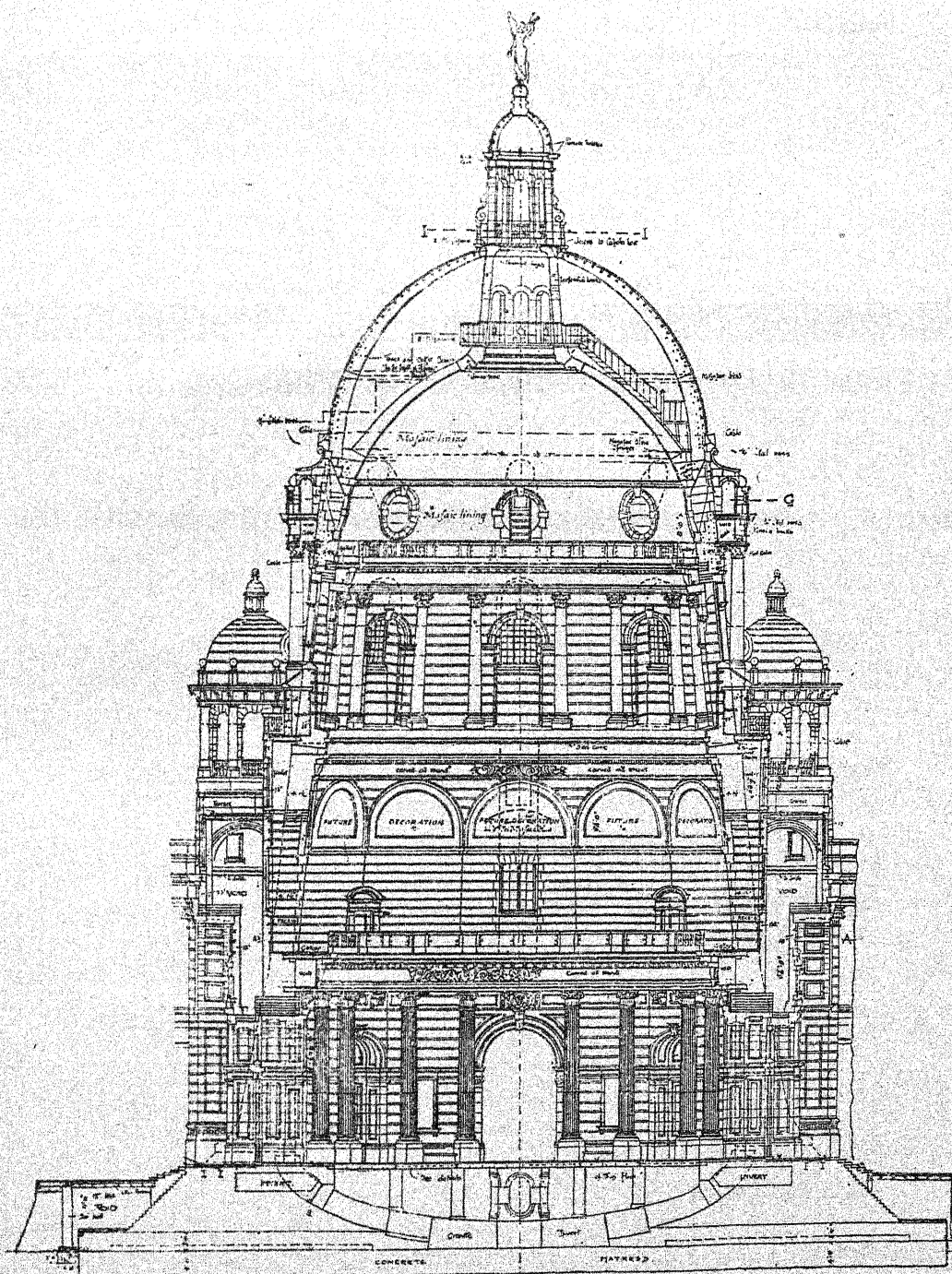
"Another remarkable feature in the building is its whispering gallery, which runs round, inside the dome, at its base. Access is gained to it from the terraced roof around the base of the dome, by eight small door ways through it. As may be seen from the section (Fig. 1) it hangs out into the building, being

supported upon the crowns of the cross arches below ; and it is about eleven feet wide, inside the low parapet wall which protects it. On entering the building a person is struck by the loud echoes which fill the place in answer to his footfall ; but these sounds are intensified on entering the gallery. The footfall of a single individual is enough to wake the sounds as of a company of persons, and, in response to ordinary conversation, strange weird sounds and mocking whispers emanate from the wall around. Loud laughter is answered by a score of friends safely ensconced behind the plaster. The slightest whisper is heard from side to side, and a conversation may be easily carried on across the diameter of the dome, in the lowest undertone, by simply talking to the wall, out of which the answering voice appears to come. A single loud clap is distinctly echoed ten times."

The foregoing description extracted from the volume on Bijapur architecture by Mr. Cousens published in the Indian Archaeological Survey series makes it clear that the whispering gallery at the Gol Gumbaz is a very remarkable one. The present writer has not yet had an opportunity of visiting it but hopes it will soon arise. It is clear that the architectural features of this whispering gallery are distinctive, situated as it is at the foot of the dome itself, instead of in a drum below it as at St. Paul's, and a fuller study of the acoustical results following from this feature would be well worth while. Judging from the case of the whispering gallery at the Victoria Memorial, Calcutta, to which reference will be found below, it is not improbable that in addition to the usual circumferential propagation of sound-waves round the gallery, there will also be found a local concentration of sound at the further end of the diameter at which the source is situated. Mr. Cousen's description indeed suggests that such an effect is present in a notable degree.

III.—THE VICTORIA MEMORIAL, AT CALCUTTA. (Plate XI.)

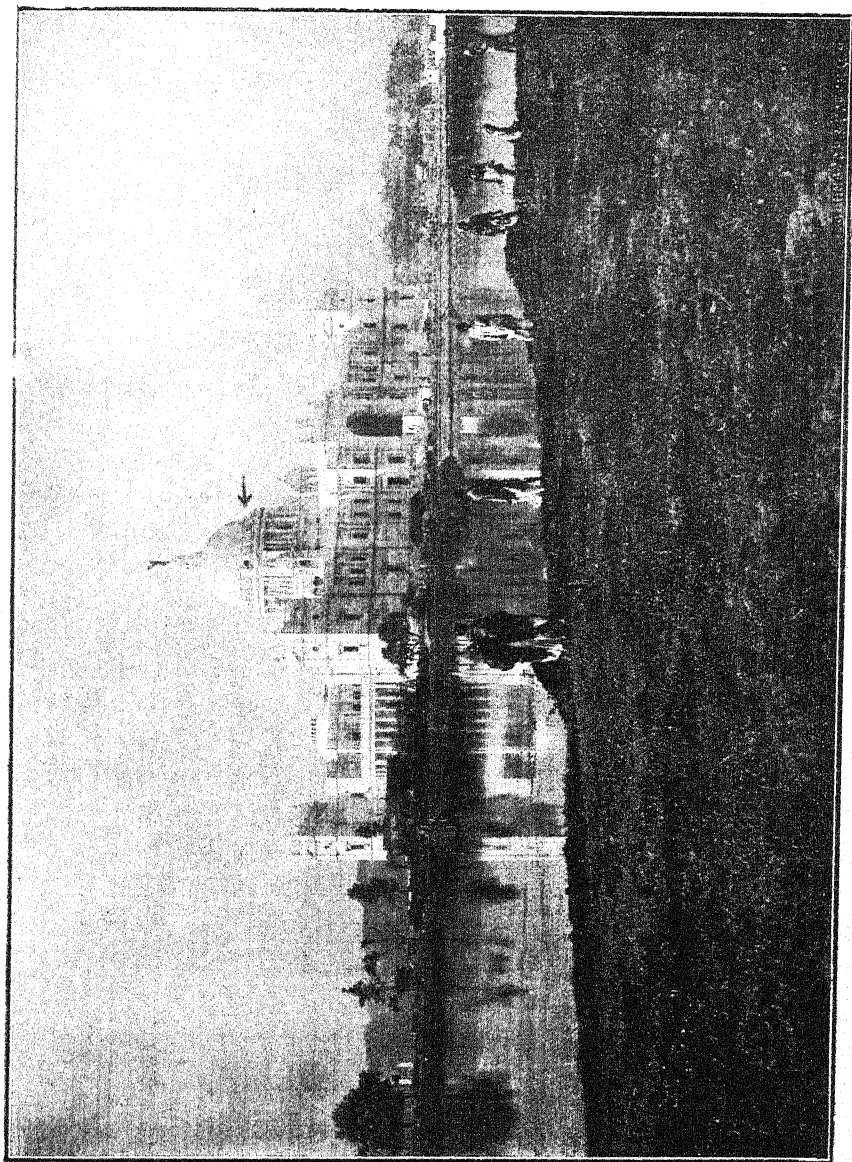
The Hindu and Moslem Rulers of India left behind them great architectural monuments which impress the imagination of the beholder, and, in not a few cases, are gems of perfection which command the admiration of the world, e.g., the Taj Mahal. Though not on the same level as these triumphs of indigenous art in greatness of conception or execution, the Victoria Memorial



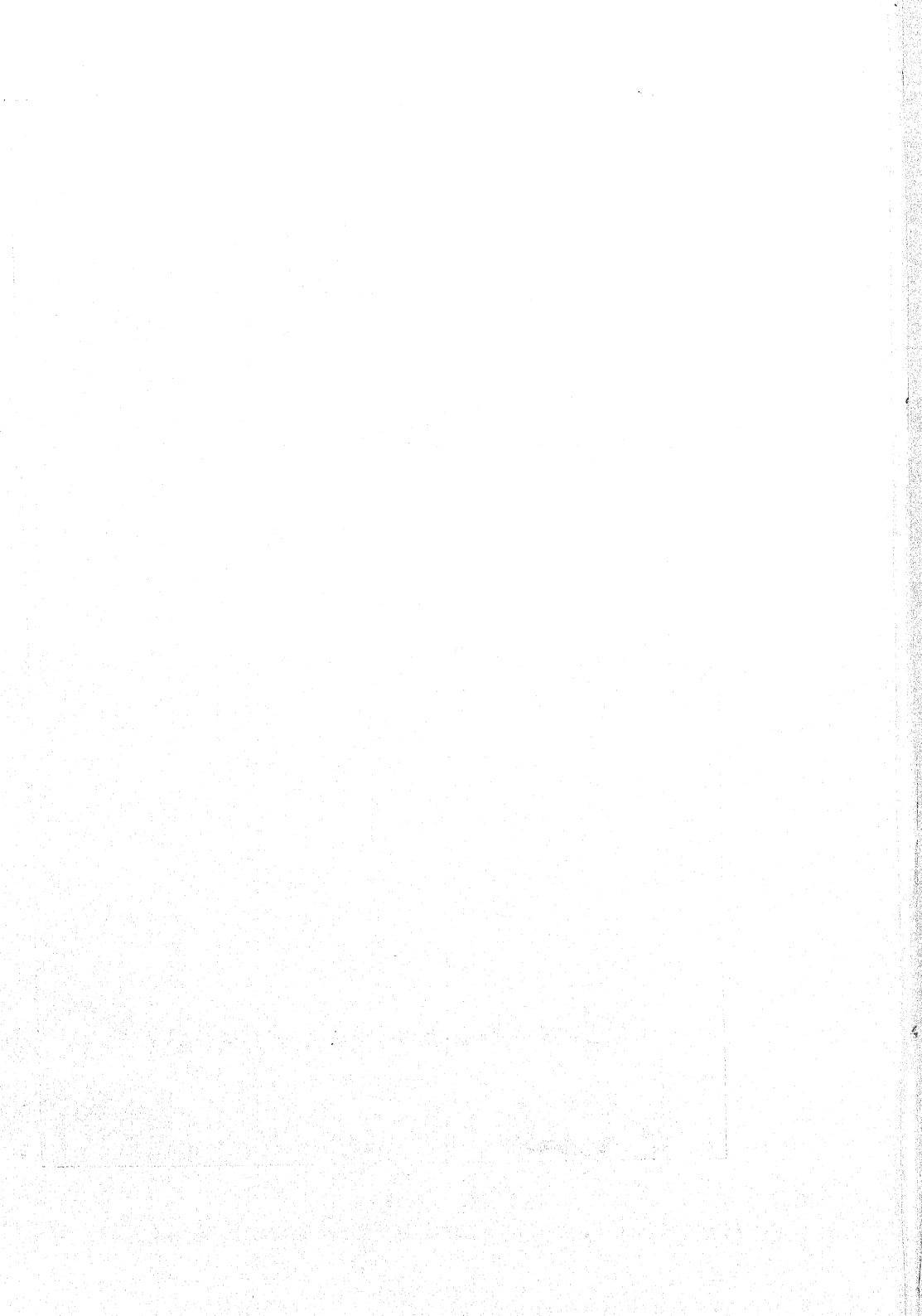
recently completed at Calcutta, may, nevertheless, claim to be the most remarkable building of any erected in India during the years of British Rule and it is certainly an ornament to the "City of Palaces." Occupying a privileged position in the Maidan or open space between Fort William and the city, this edifice of white Indian marble with its dome and winged statue of victory crowning all, catches the eye from afar and is a worthy monument of the reign of the Queen whose memory it seeks to perpetuate. The building took twelve years to construct and cost over half a million sterling. It is intended to be a treasure-house for historical paintings, sculptures, and other relics, and stands in extensive grounds which are being laid out as a public garden with ornamental tanks and bridges.

Going up by the grand stair-case facing the Maidan towards the Ochterlony monument, and passing through the portico and entrance room, the visitor finds himself in the circular Queen's hall which stretches up from the floor to the base of the inner dome of the building. In the centre of this stands the statue of the young Queen Victoria. Some 35 feet up from the floor is a gallery with its walls forming a dodecagon and above this, on the wall, twelve semi-circular spaces covered with paintings representing her life. Still higher, some 95 feet from the floor, is a second circular-gallery four feet broad which lies just at the base of the inner dome and is surrounded by a marble railing. The circular wall of the gallery here has a very decided slope inwards and is interrupted some few feet above the floor of the gallery by seven great circular windows which are visible from outside and by the opening for the stair-case by which admission to this gallery is obtained. An opening is also provided above by which it is easy to enter the space between the two domes and pass round it by a circular foot-path. The inner dome is open at the top to which access can be obtained by stairs. These features are indicated in the architectural section in Fig. 2.

The circular gallery at the base of the inner dome, and the space between the two domes form two very perfect whispering galleries, the former of which was first discovered by the writer. The diameter of the former is 39 feet and of the latter 56 feet. Their special acoustical features have been studied by the writer and will be referred to more fully in Section V below.



The Victoria Memorial at Calcutta.



IV.—THE GRANARY AT BANKIPORE (PATNA). (See Plate XII.)

"At once the most prominent and the most curious building in Bankipore is the old Government Granary known as the Gola. This is a brick building, 96 feet high with walls 12 feet thick at the bottom, built in the shape of a beehive or half an egg placed on end, with spiral two stair-cases on the outside winding to the top; it is said that Jang Bahadur of Nepal rode on horse-back up one and down the other. This dome shaped structure was erected sixteen years after the great Famine of 1770, as a store-house for grain, it being intended that the grain should be poured in at the top and taken out at the bottom through the small door there. Owing to a curious mistake on the part of the builders, these doors were made to open inwards. The following inscription is on the outside :—

"No. 1. In part of a general plan ordered by the Governor-General and Council, 20th of January, 1784, for the perpetual prevention of Famine in these provinces, this Granary was erected by Captain John Garstin, Engineer. Completed the 20th of July, 1786. First filled and publicly closed by——."

The store-house has never been filled and so the blank in the inscription still remains, while the opening at the top is closed by a great stone slab. It stands to this day the monument of a mistake. During the famine of 1874, a quantity of grain, which, if left at the railway station might have been injured by the rain was temporarily stored there, and in times of scarcity proposals are still made to fill it with grain. But the loss from damp, rats and insects renders such a scheme of storing grain wasteful and impracticable. This building, once intended to meet the requirements of the whole district in time of famine, is now only useful as a store-house for furniture. It is chiefly remarkable for its reverberating echo, which answers to the slightest sound, a whisper at one end being repeated at the other. It is a landmark for a considerable distance along the river and commands a fine view of the surrounding country."

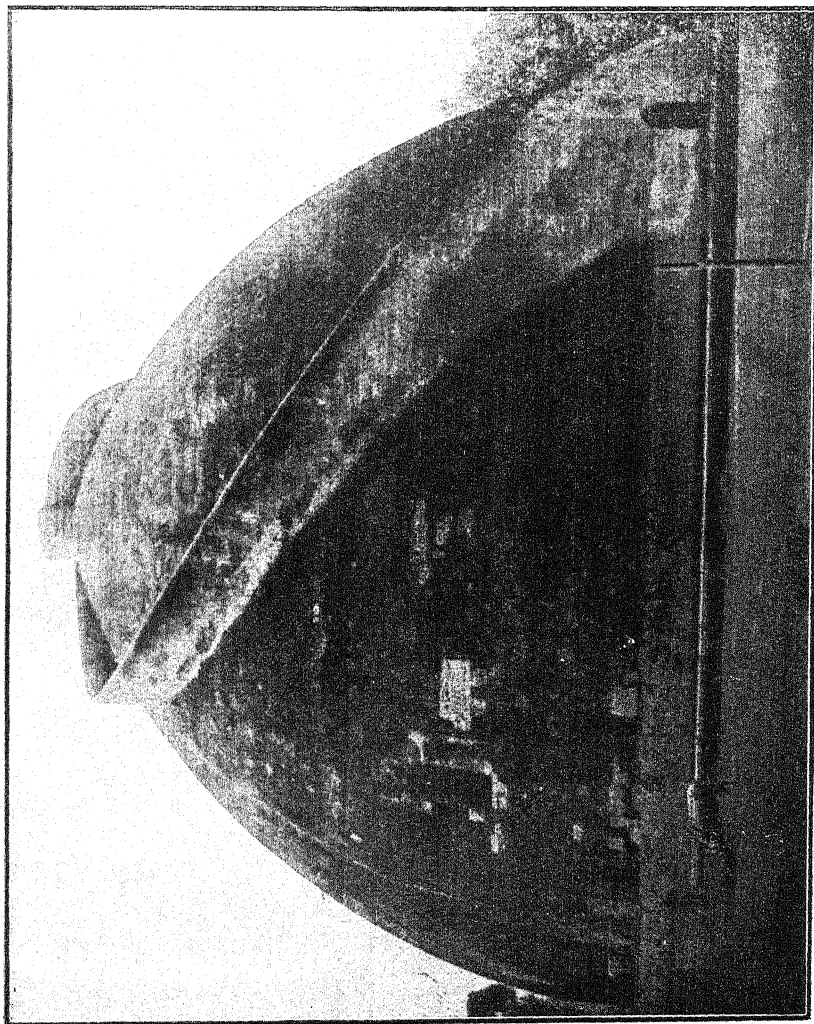
The foregoing description is taken from the Patna District Gazetteer. In September 1922, the writer paid a brief visit to Bankipore and looked over this building, but had no time to make a thorough scientific examination of its acoustics. This is obviously

a task that ought to appeal to the physicists at the local University of Patna. The most striking acoustical feature that was noticed by the writer was the return of the sound from the walls of the building as a surprisingly loud single echo when the observer stood at its centre and uttered a syllable or two. This was evidently due to the curved walls acting as a concave mirror focussing the sound at the same point as its origin. When the observer moved away from the central position, the simple return of sound gave place to a multiple echo.

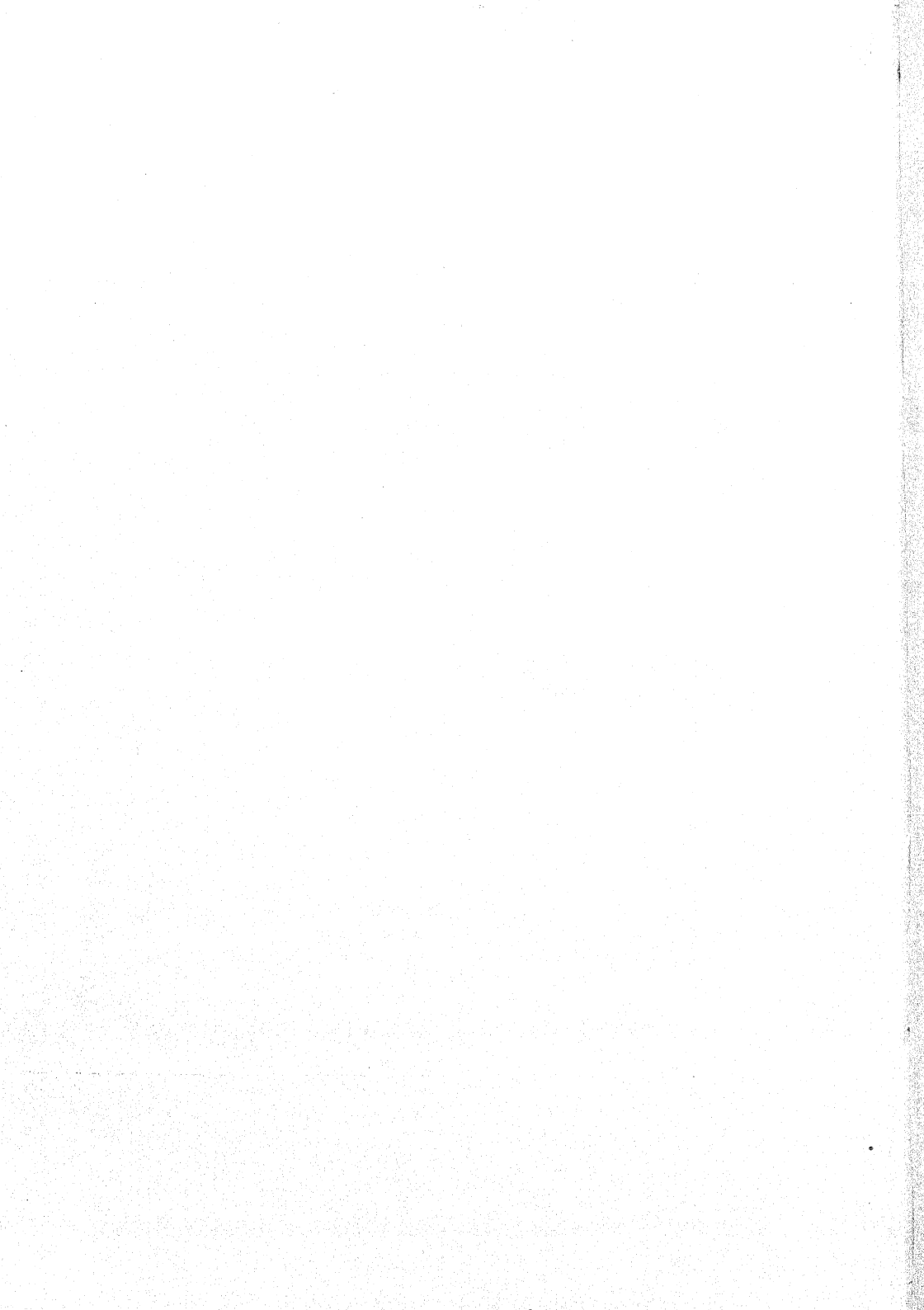
The interior of the building is rather gloomy, as it is lighted only through the four doors in its massive brick walls. Inside, the brick work is bare, and has not been plastered over. The surface of the wall is thus not particularly smooth. An attempt to carry on a conversation in a low undertone with another observer situated 90° off along the curved wall was only partially successful. A further thorough study of the acoustics of this structure would be well worth while in order to explain the formation of the curious echoes heard in it.

V.—WHISPERING GALLERY AT THE CALCUTTA GENERAL POST OFFICE. (See Plate XIII.)

This gallery whose acoustical properties were first discovered by the writer merits a brief description. The General Post Office is the most imposing building amongst the many stately piles that surround Dalhousie Square at Calcutta. It is crowned by a dome set on a high cylindrical drum, the upper part of which is occupied by a row of windows which illuminate the building; the lower part of the drum consists of a perfectly vertical smooth circular wall some fifteen feet high. This is provided with a gallery four feet broad to which admission is obtained through a single door, which opens in from the terrace of the building. The diameter of the gallery is 57 feet. Immediately below the gallery is the public rotunda. Unfortunately owing to the location of the building and one side of the rotunda being open towards Dalhousie Square, the drum of the dome is full of the hum of public traffic throughout the day. But early in the morning or late in the evening when the bustle of traffic has died away, the whispering gallery is fully worth a visit by the interested student of physics. It shows effects similar to those observed at St. Paul's though not in such a high degree.



The Granary at Bankipore.



VI.—PROPAGATION OF SOUND IN WHISPERING GALLERIES.

The comparative study of the three whispering-galleries at Calcutta has yielded result of interest. The three galleries are approximately of the same size; the differences in their architectural characters are however considerable, and the differences in their acoustical characters consequent thereon are quite distinct. The gallery at the Calcutta General Post Office is architecturally very similar to that at St. Paul's in London, though smaller in size, with the important difference that its walls are perfectly vertical while those of the St. Paul's gallery slope distinctly inwards as has been pointed out by Prof. Sabine in his article. The fact that the effects observed in the latter are clearly more striking appears to support Prof. Sabine's contention that the inward slope of the wall is an important feature contributing to the efficiency of the whispering gallery.

The lower gallery at the Victoria Memorial whose acoustical features were first observed and studied by the writer has quite distinctive features of its own. This gallery is just below the inner shallow dome, and its wall is broken by eight large openings. In fact the continuous part of the wall above the floor of the gallery is only three feet high. Nevertheless, the gentlest whisper at any part of the wall is heard right round the gallery, particularly if the observer and his assistant stoop down a little towards the floor. The effect is hardly less striking than that observed at St. Paul's. As the wall of the gallery has a very marked slope inwards, this appears to furnish further support for Prof. Sabine's views. Part of the effect is however doubtless due to the presence of the curved surface of the dome above, of which indeed the wall of the gallery practically forms a part. If the observer and his assistant both stand up to their full height, and face each other directly, it is distinctly easier to converse in an undertone when they are at the opposite ends of a diameter than when they are a smaller distance apart.

The upper gallery at the Victoria Memorial which lies between the two domes naturally shows very striking effects, owing to the comparatively enclosed character of the space; the ease with a whisper any where is heard throughout is distinctly uncanny, particularly as the observer and his assistant are hidden from each other by the mass of the inner dome. As will be seen from the

architectural section, the wall of the gallery slopes pretty steeply inwards even at the lowest point.

Experimenting in these whispering galleries, the existence of circumferential and radial nodal lines in the acoustical field due to a steady source of sound was established as in the case of the St. Paul's gallery. Another interesting point to which the writer has not seen attention previously directed is the study of the propagation of sounds of an *impulsive* character in whispering galleries. Visitors to St. Paul's notice immediately the peculiar multiple echoes which accompany the sound of footfalls in the gallery; a single hand-clap is repeated five or six times. The echoes appear to proceed from somewhere near the opposite point of the wall of the gallery. Similar multiple echoes are noticed in the Calcutta galleries; the effect is least marked in the General Post Office gallery, much better in the lower gallery at the Victoria Memorial and appears in an extraordinarily exaggerated form in the upper gallery between the two domes, a single hand-clap or other sharp sound produced in the gallery being heard repeated no fewer than twenty times. As the observer producing the sound mounts the stair towards the top of the inner dome, the effect becomes less and less striking and ultimately vanishes at the centre. These observations give the clue to the real nature of the phenomenon; the multiple sound is not due to any echo, but is merely due to the fact that a sound-wave generated at any point on the gallery travels circumferentially round and round the gallery many times and is heard each time it passes the observer before it finally ceases to be audible. The smaller the decrement of intensity between two successive returns, the greater is the efficiency of the whispering gallery. The character of the sound at each successive return also undergoes a distinct alteration, as of course is to be expected owing to the differing decrements for sounds of different pitch.

The foregoing simple explanation of the multiple sounds heard in whispering galleries was verified in two ways. Firstly, if the origin of the sound and the observer are situated at different points in the gallery, the sound waves moving in opposite directions would pass the observer at different instants, and two sets of multiple sounds should thus be heard except when the observer and the source are exactly at opposite ends of a diameter. This

was actually found to be the case, and the relative intensities of the sounds and the points of the gallery from which they appeared to emerge showed curious variations as the position of the observer was shifted.

The second method of verification was to determine the time interval between successive returns of the sound. With a little practice, this could be done in all the three galleries and was a particularly easy task in the upper gallery of the Victoria Memorial. By giving a tap periodically say at each tenth return of the sound, the succession could be kept up indefinitely, and the time taken for a few hundred returns of the sound could be determined with a stop-watch. The interval between successive returns of the sound was found to be equal to the circumference of the gallery divided by the velocity of sound, correct to within one per cent. It would be interesting to repeat this experiment in the larger galleries at St. Paul's and the Gol Gumbaz at Bijapur.

VII.—SYNOPSIS.

The paper describes five whispering galleries : (1) the Gol Gumbaz at Bijapur ; (2) the upper gallery between the two domes of the Victoria Memorial at Calcutta ; (3) the lower gallery under the inner dome of the Victoria Memorial ; (4) the Granary at Bankipore ; and (5) the gallery at the Calcutta General Post Office. Of these (3) and (5) were discovered by the writer, and in quality, (3) is not greatly inferior to the gallery at St. Paul's at London.

The comparative study of (2), (3) and (5) has led to some interesting results, notably the confirmation of Sabine's view of the importance of an inward slope of the wall of the gallery for giving the best effects, and the concentration by a spherical dome of a maximum of sound at the opposite end of the diameter. Circumferential and radial nodal lines were observed in these galleries similar to those observed at St. Paul's. Further, the study of the propagation of impulsive sounds in these whispering galleries showed that the multiple sounds heard are not echoes as might be thought at first, but are due to the fact that the sound-wave travels circumferentially round the gallery several times before it is sensibly extinguished, and is heard each time as it passes the observer. The smallness of the decrement in successive returns is a measure of the strength of the whispering gallery effect. The

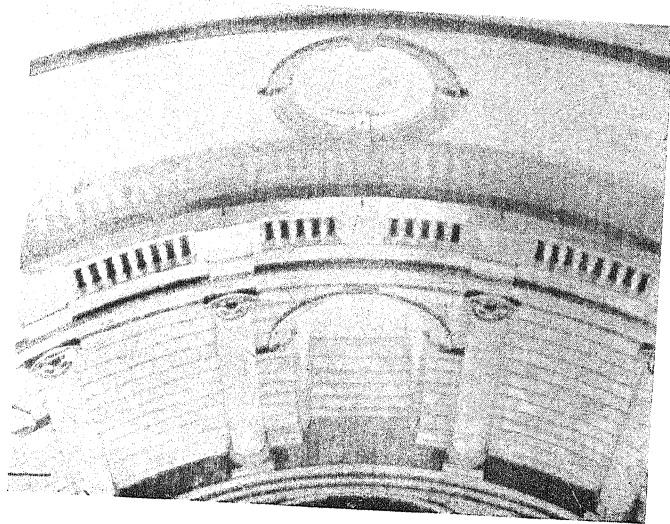
interval between successive returns is equal to the circumference of the gallery divided by the velocity of sound within an accuracy of one per cent. The waves travelling in opposite directions round the gallery can be differentiated by ear.



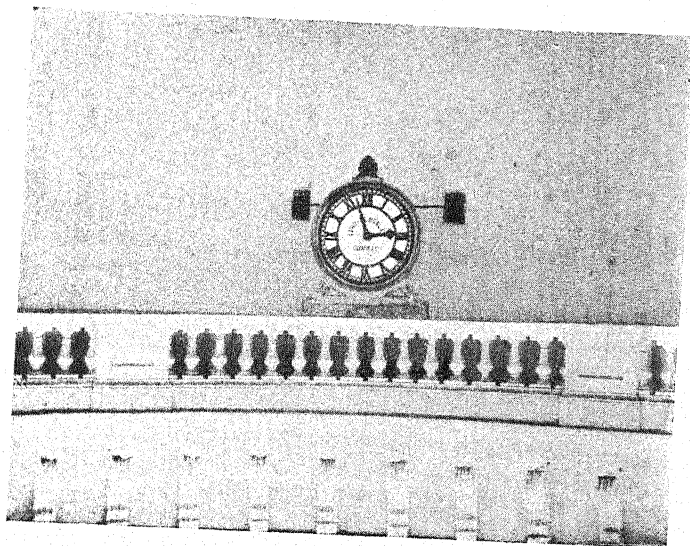
The Calcutta General Post Office.

RAMAN.

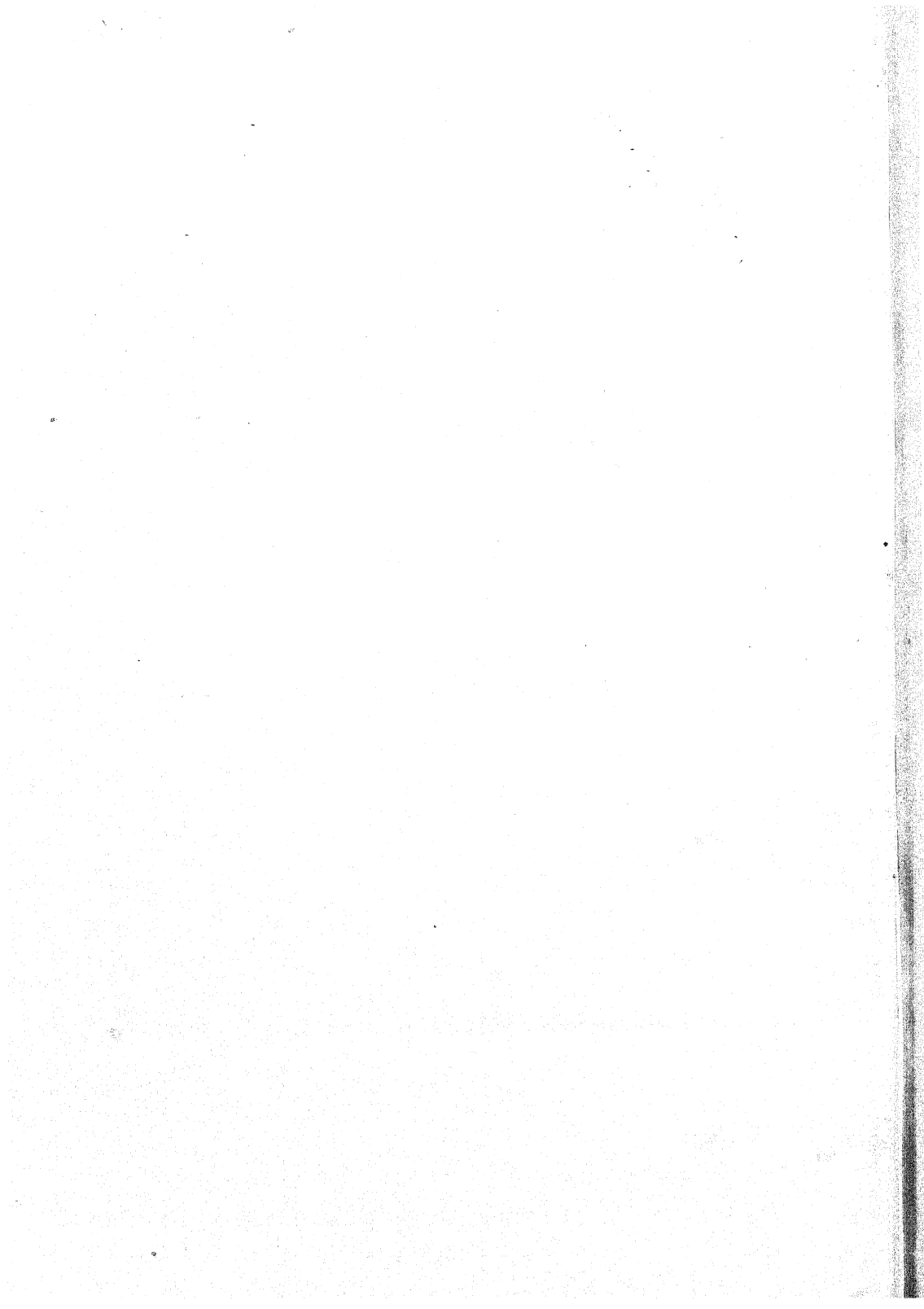
PLATE X.V.



Part of circular gallery at the Victoria Memorial.



Part of circular gallery at the Calcutta General Post Office.



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